Classical Electrodynamics

Alexander Altland

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Chapter 1

Introduction

1.1 History

We begin with a brief chronological account of the development of electrodynamics:

1.1.1 Ancient world and middle ages

Ancient world: Unlike with mechanics/astronomy, there is no 'theory' of electromagnetism. Known phenomena include: the attraction of wool and other types of tissue by rubbed amber (greek $\rightarrow$ electron) and the attraction of metals such as iron by magnetite ('magnesia' $\rightarrow$ city in ancient Turkey).

Middle ages: Petruus Peregrinus (Pierre der Maricourt), 1269: applies experimental methods (!) to measure the forces exerted by a spherical magnet. Coins the term **poles** of a magnet.

1.1.2 Modern times and age of enlightenment

In his work 'de magnete', 1600, **William Gilbert**, 1544–1603 presents a thorough analysis of **magnetic phenomena**. Topics covered in his work include the magnetic nature of the earth, the character of the forces between magnets ($\rightarrow$ torsion), a theoretical understanding of compasses, and the fundamental differences between electric and magnetic forces (attractive forces, 'attractio' vs. torsional forces 'verticitas').

In the 17th century, not much progress is made (Mechanics is considered more interesting.) However, electric devices (bases on mechanisms of friction) become popular as tool of entertainment at european courts.

Early 18th century: the era of 'qualitative electrostatics': **Benjamin Franklin**, 1706–1790, invents the **lightening rod** and introduces the notion of **electric charge** (a term inspired by military vocabulary, the 'discharging' of a canon, similarly with 'batteries'.)[1] The existence of positive and

[1] Arguably Franklin owed much of his success to the fact that he lived in America and remained, therefore, largely uninfluenced by false concepts circulating in Europe.

negative charges is understood.

Late 18th century: the era of 'quantitative electrostatics': **Charles Auguste de Coulomb**, 1736–1806, discovers the **Coulomb force** $\mathbf{F} \sim q_1 q_2 \mathbf{r}/r^3$ between electric charges. (Independently discovered by others.) Much underrated, **Henry Cavendish**, 1731–1810: discovers Coulomb law, understands essential aspects of the physics of dielectric materials and of electric conduction. (But is a lazy publisher.)

Alessandro Volta, 1745–1827, invents the Voltaic column (and presents it to Napoleon), emancipation of unreliable friction devices.

George Green (date of birth unknown) and **Karl Friedrich Gauss** (1777–1855) develop the 'final form' of electrostatics.

1.1.3 Romantic era till late nineteenth century

Early 19th century: the era of 'magnetostatics': the invention of the electric battery paves the way to the systematic analysis of electric currents and to conduction experiments.

Inspired by the romantic idea of a 'global spirit' and the understanding that all forces are mere manifestations of a unified force (Schelling), researchers such as Oersted and Faraday attempt to understand electricity and magnetism on a common basis. Nature philosophy actively influences science!

Hans Christian Oersted, 1777–1810, qualitative discovery of magnetic forces induced by electric currents. (Quantitative theory → Biot and Savart.) The advent of Electro–Magnetism. **Anrdré Marie Amperè**, 1775-1836, discovery of forces between conductors carrying currents. **Georg Simon Ohm**, 1789–1854, discovery of Ohm's law.

Michael Faraday, 1791–1867, seeks for reversal of Oersted's experiment ('convert magnetism into electricty') → the law of induction, principle of the generator. Faradays work leads to the emergence of a fundamentally new understanding of electromagnetic forces: the idea of a **field** (→ a conceptual revolution), i.e. the understanding that, independent of the existence of matter, fields are an intrinsic quality of space. Matter influences fields and vice versa. However, the existence of fields is not tied to the presence of matter. Essentially, this picture forms the basis of our contemporary understanding of all forces.[2]

James Clerk Maxwell, 1831–1879, Maxwells equations of electromagnetism (1862). Fields acquire own existence (independent of matter) and governed by their own fundamental laws.

Heinrich Hertz, 1857–1894, discovery of electromagnetic waves (1887) → first transatlantic radiowaves 1901. Unification of electrodynamics and optics.

[2]Faraday tried to include gravitation into the unified picture, a work that is still far from completion!

1.1.4 Twentieth century

Albert Einstein, 1879–1955, theory of special relativity (1905). (Work inspired by earlier insight of **Hendrik Lorentz**, 1853–1928.)

Realization that electrodynamics is not a closed theory (difficulties with point charges.) → embedding of electrodynamics in Quantum electrodynamics (developed as of 1929.)

1.2 The apparatus of electrodynamics

1.2.1 Basic quantities

Def.: **(classical)**[3] **electrodynamics**: Classical theory of electric and magnetic phenomena.

Central to the theory of electrodynamics ED: four **electromagnetic fields**:

field	denotation (historical)
$\mathbf{E}$	electric field strength
$\mathbf{B}$	magnetic induction
$\mathbf{D}$	electric displacement
$\mathbf{H}$	magnetic field

Table 1.1: Electromagnetic fields

▷ Info. Mathematically, a **field** is a mapping $\phi : M \to T, x \mapsto \phi(x)$ from a base manifold M into a target manifold T.

Examples: Temperature distribution in the lecture hall, a mapping (lecture hall)$\subset \mathbb{R}^3 \to \mathbb{R}, \mathbf{r} \mapsto T(\mathbf{r})$ — a scalar field in three–dimensional space. Wind weather forecast, a mapping (region of forecast)$\subset \mathbb{R}^2 \to S^1, \mathbf{x} \to \mathbf{n}(\mathbf{x})$, where the unit vector $\mathbf{n}(\mathbf{x})$ defines the direction of wind — a field on the unit circle, S^2, etc.

The fields $\mathbf{X} \equiv \mathbf{E}, \ldots, \mathbf{H}$ are vector fields in (3+1) dimensional space–time:[4]

$$\begin{aligned} \mathbf{X}: \quad \mathbb{R}^3 \times \mathbb{R} &\to \mathbb{R}^3, \\ x \equiv (\mathbf{x}, t) &\mapsto \mathbf{X}(x). \end{aligned}$$

▷ Info. In fact, the fundamental fields of ED are *not* vector fields! The identification $\mathbf{E}, \ldots, \mathbf{H} \leftrightarrow$ (vector fields) is an outgrow of history. It makes sense only locally, and as long as we stick to a fixed

[3] As with classical mechanics, classical electrodynamics (ED) is not a closed theory. It runs into problems at small length scales (→ infinite self energies) which find their solution within the larger framework of **quantum electrodynamics**.

[4] Notation: Most vectors in ED are defined in $3 + 1$ space–time dimensions. The space like components will be denoted by boldface characters, e.g., $\mathbf{x} \in \mathbb{R}^3$. The unification with the time–like component (e.g., t) to a four–component object will be often be denoted by a non–boldface symbol, e.g., $x = (t, \mathbf{x}) \in \mathbb{R}^4$.

intertial system. (cf. with the phase space of classical mechanics which, also, is not a vector space.) The reason why we stick to the vector field convention in the early parts of this course is its overwhelming prevalence in the physics community.

To complete the setup of the theory, we need to introduce two more fields, the **matter fields** of ED:

matter field	denotation
ρ	charge density
$\mathbf{j}$	current density

Table 1.2: Matter fields

The **charge density** is a scalar field $\rho : \mathbb{R}^4 \to \mathbb{R}, x \mapsto \rho(x)$ describing the distribution of charges in space and time (see section 1.2.4 below.) the **current density** $\mathbf{j} : \mathbb{R}^4 \to \mathbb{R}^3, x \mapsto \mathbf{j}(x)$ is a vector field describing the distribution of electric current (densities).

1.2.2 Interaction of matter and fields

Charge and current density act as 'sources' driving the generation of electromagnetic fields. The connection (matter fields $\rightsquigarrow$ electromagnetic fields) is provided by the celebrated **Maxwell equations** (expressed in Gaussian units, see section 1.2.3 below)

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \tag{1.1}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D} = \frac{4\pi}{c}\mathbf{j}, \tag{1.2}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0, \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.4}$$

where $c = 3 \cdot 10^8$ m s^{-1} is the vacuum speed of light. Eqs. (1.3) and (1.4) are the homogeneous (source–free) equations, Eqs.(1.1) and (1.2) the inhomogeneous Maxwell equations.

The Maxwell equations alone do not suffice to uniquely determine the dependence of the fields $\mathbf{E}, \ldots, \mathbf{H}$ on the sources $j \equiv (\rho, \mathbf{j})$. (Too few differential equations for $4 \times 3 = 12$ field components!) Implicit to the Maxwell equations are relations $\mathbf{D} = \mathbf{D}[\mathbf{E}]$ ($\mathbf{H} = \mathbf{H}[\mathbf{B}]$) determining the electric displacement in terms of the electric field strength (the magnetic field in terms of the magnetic induction.)[5] Particularly simple connections $\mathbf{D} \leftrightarrow \mathbf{E}$ ($\mathbf{H} \leftrightarrow \mathbf{D}$) prevail in the case of vacuum electrodynamics (on which we will focus in the first chapters of this course):

$$\text{vacuum:} \qquad \mathbf{D} = \epsilon_0 \mathbf{E}, \qquad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B}, \tag{1.5}$$

where the **dielectric constant** $\epsilon_0 = 1$ and the **magnetic permeability** $\mu_0 = 1$ are constants,[6] i.e. up to constants of proportionality (whose significance will be discussed below), the fields $\mathbf{D}$ and $\mathbf{E}$

[5] For two arbitrary functions f and g, the notation $f[g]$ indicates that f is a **functional** of g, i.e. depends on the function g as a whole. E.g. the notation $\mathbf{D}[\mathbf{E}]$ indicates that, in general, $\mathbf{D}(\mathbf{x}, t)$ may depend on the values $\{\mathbf{E}(\mathbf{x}', t')\}$ takes at different instances of space and time.

[6] ... assuming the value unity only in the Gaussian system of units chosen here, cf. section 1.2.3 below.

($\mathbf{H}$ and $\mathbf{B}$) can be identified with each other.

▷ Info. From a purist point of view the sources of Maxwells equation contain all charges and currents in the universe. In particular the abundance of microscopic charges and currents bound in extended matter are explicitly included in the source terms. Within this **microscopic or vacuum formulation of electrodynamics** one may identify $\mathbf{E} = \mathbf{D}$ and $\mathbf{H} = \mathbf{B}$. In general, however, the microscopic approach is met with insurmountable problems: *Conceptually*, the correct description of charges on microscopic scales requires a quantum mechanical formulation, i.e. a theoretical framework beyond the scope of classical electrodynamics. *Practically*, it is impossible to solve Maxwells equations for, say, the $\mathcal{O}(10^{23})$ charges embodied in a macroscopic piece of matter. This means that the microscopic form of Maxwells equations is applicable only to those situations where the system under consideration is effectively matter–free (except for a few isolated carriers of charges and currents that may be described in the idealized manner discussed in section 1.2.4 below.) The extension of the theory to the presence of extended matter will be discussed in chapter xxx below.

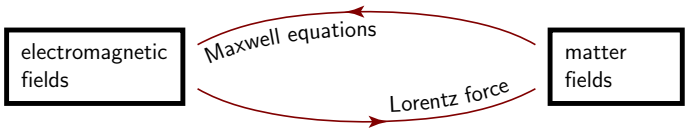


Figure 1.1: On the mutual influence (electromagnetic fields ↔ matter fields)

For a given dependence $\mathbf{D}[\mathbf{E}], \mathbf{H}[\mathbf{B}]$ (and given boundary conditions, see below) the Maxwell equations uniquely specify the fields in terms of the charges. Reciprocally, the fields exert influence on the charge distribution. Responsible for this feedback mechanism is the **Lorentz force (density)**

$$\boxed{\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B},} \tag{1.6}$$

where $\mathbf{f}(\mathbf{x}, t)dV$ is the force a small element of matter concentrated in the volume element dV experiences due to the presence of the electromagnetic fields. Thus, *electromagnetic fields are influenced by matter fields and vice versa.*

▷ Info. Consequence of the mutual influence of fields on matter: a new understanding of **forces between particles**. Newtonian mechanics: the interaction between two (charged) particles A and B is instantaneous $\rightsquigarrow$ contradiction with the principle of relativity, i.e. the existence of a maximal speed of signal propagation (the velocity of light), see below. In ED, the interaction between A and B is mediated by a field: the charge of particle A creates an electromagnetic field (via the Maxwell equations) which then influences B (via the Lorentz force.) This indirect interaction is retarded (by the speed of field propagation which will turn out to be the speed of light.) History: the experimentally observed forces between charges could not be understood in terms of an instantaneous Newtonian interaction and *required* the introduction of non–matter fields into our description of nature. Modern physics describes all four fundamental iqnteractions as indirect processes matter ↔ (gauge) field ↔ matter.

Indeed, we will see that ED is compatible with the principles of special relativity while classical Newtonian mechanics is not. Precise formulation: under a general Lorentz transformation (see chapter xx), the

Maxwell equation remain form invariant. However, this transformation is not manifest (a consequence of the interpretation of the EM fields as vector fields!)

1.2.3 Maxwell equations: generalities

Objectives of this lecture course:

- ▷ Understand connection matter fields $\rightsquigarrow$ EM fields, i.e. explore how to solve the Maxwell equations. (Roughly two thirds of the course.)
- ▷ Understand the general structure of the theory $\rightsquigarrow$ special relativity and its ramifications in ED.

Here: warmup to the first task.

Structure of the Maxwell equations

Maxwell equations: a system of *coupled first order linear partial differential equations*. (The good news: 'first order' and 'linear'). Apart from problems with exceptionally high symmetry no closed solutions can be found (cf. with situation in Newtonian mechanics.) $\rightsquigarrow$ numerical integration theory of Maxwell equations in engineering. However, a far reaching understanding of the general behaviour of fields can be developed, a consequence of the fact that

Electrodynamics is a linear theory.

Which means that if $\mathbf{E}, \ldots, \mathbf{H}$ $(\mathbf{E}', \ldots, \mathbf{H}')$ solves the Maxwell equations for sources $(\mathbf{j}, \rho)$ $((\mathbf{j}', \rho'))$, the fields $\mathbf{E} + \mathbf{E}', \ldots, \mathbf{H} + \mathbf{H}'$ will be a solution to $(\mathbf{j} + \mathbf{j}', \rho + \rho')$, the **superposition principle**. For example, to understand the electric field created by a system of point charges in space, it suffices to understand the field created by a *single* point charge. The field of the full charge distribution obtains by adding the constituent fields, etc.

Units

The detailed form of the equations of ED depends on the chosen system of units. Two principal approaches:

- ▷ The practitioners point of view: Stick to the international system of units (**SI–system**, aka **MKSA–system**) wherein length (1 m), mass (1 kg), time (1 s), *and* current (1A) carry their own physical units. Express all equations so as to conform with this unit system.
- ▷ The purists point of view: avoid the introduction of new units as much as possible and use the **Gauß system** aka **cgs–system** of units. For this, I am willing to pay the price of being at odds with most of the people (engineers & applied physicists) working with electromagnetic phenomena.

The SI–system is very lab–friendly — an invaluable advantage. Yet by introducing units which are only seemingly fundamental (1A), it tends to obscure the principal connection between physical quantities. For this reason *we will use the Gauß system of units in this course*.

▷ Info. The **definition of the Ampère** in the SI–system. Two infinite parallel wires kept at a distance of 1m carry a uni–directional electric current of 1A each if they exert an (attractive) force of $2 \cdot 10^{-7}$ N on each other. The introduction of the Ampère determines the **unit of electric charge (SI–system)** as 1 C $\equiv$ 1 A·s.

In addition to the Ampère, the SI–system contains a number of secondary (non–fundamental) units. Specifically, the strength of the electric field is measured in volt/meter: A point in space is subject to an electric field of strength one **volt** per meter (1 V/m) if a charge of 1C placed at that point will experience a force of strength F=1N=1 m kg/s^2. Since the electrostatic Lorentz force is $F = QE$, we find that 1V = 1 m^2 kg /(A s^3). The Maxwell equation $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ implies $[\mathbf{B}] = [\mathbf{E}] \times$ (time/length). Magnetic fields are measured in units **tesla**, where 1T = 1 V s/m^2.

The **Maxwell equations in the SI–system** assume the form

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} &= \mathbf{j}, \\ \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \end{aligned}$$

where the value of the vacuum dielectric constant is $\epsilon_0 = 8.8544 \cdot 10^{-12}$ As/Vm and the magnetic permeability is given by $\mu_0 = 12.5664 \cdot 10^{-7}$ Vs/Am.

To elucidate the **difference between the two systems** consider two static charges q_1 and q_2 at distance r. Anticipating our discussion below we note that the charge q_1 (kept at $\mathbf{r} = 0$, say) will create a static $\mathbf{D}$–field of strength $D(\mathbf{r}) = c_0 \cdot q_1/r$, where $r = |\mathbf{r}|$ and the constant $c_0 = 1$ (Gauß) $c_0 = 1/4\pi$ (SI) depends on the convention chosen in the first of the Maxwell equations. This corresponds to a field $\mathbf{E}$ of strength $E = \epsilon_0^{-1} D = c_0 q_1/(\epsilon_0 r^2)$. By virtue of the Lorentz law the second charge will then feel a (Coulomb) force $F = c_0 q_1 q_2/\epsilon_0 r^2$.

According to the philosophy of the Gauß system, this law — bridging between mechanic and electromagnetic quantities — *defines* the unit of charge. Specifically, we say that two point particles carry a charge of **1 esu** (one electrostatic charge unit) if they experience a mutual force of 1 dyn = 1 g cm/s^2 when kept at a distance of 1cm: 1 esu = 1 g$^{1/2}$ cm$^{3/2}$ s^{-1}. (Units for magnetic fields, currents, etc. are introduced in a similar spirit and will be discussed whenever we need them.) In contrast, the SI–system has already fixed the unit of charge by the redundant yet convenient introduction of the Ampère. Experimentally, one finds that two charges of 1C each kept at a distance of 1m exert a force of strength F=8.987 $\cdot$ 10^9 N on each other, i.e. 8.987 $\cdot$ 10^9 N = $\frac{1}{4\pi\epsilon_0}$ 1 Q^2 m^{-2}. Solving for ϵ_0, we obtain the value given above. Thus, the price to be payed for the introduction of handy units of charge is the appearance of inconvenient conversion factors between the fields.[7] In the CGS system there is no need to distinguish between the electric field and the displacement field and we may set $\epsilon_0 = 1$. Relatedly, the SI–system obscures the connection between quantities which, fundamentally, carry the same physical dimension.

[7] Similar reasoning can be applied to explain the value of the magnetic permeability.

For an in–depth discussion of systems of units (including a few more than those discussed here), we refer to [1].

1.2.4 Charges and currents

General definition of charge and current distributions

We begin our analysis of the Maxwell equations with a discussion of its sources.

▷ Math. To prepare our discussion below, we recapitulate **the laws of Stokes and Gauß**, and introduce a few more integral theorems of vector analysis.

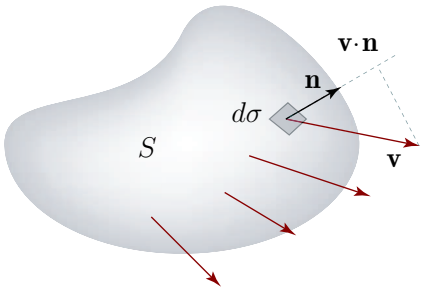


Figure 1.2: Schematic on Gauß's law

Gauß' law: Let S be a smooth orientable and closed surface embedded in $\mathbb{R}^3$. Let $\mathbf{v} : U \to \mathbb{R}^3$ be a vector field where the support $U \supset V(S)$ and $V(S)$ denotes the volume enclosed by S. Then

$$\boxed{\int_S d\sigma\, \mathbf{v}\cdot\mathbf{n} = \int_{V(S)} d^3x\, \nabla\cdot\mathbf{v},} \tag{1.7}$$

where $\mathbf{n}(\mathbf{x}\in S)$ is a unit ($|\mathbf{n}|=1$) and outwardly directed vector at the point of the surface element of S.

▷ Example: Consider a radially symmetric vector field $\mathbf{v}(\mathbf{r}) = f(r)\mathbf{e}_r$, where $\mathbf{e}_r$ is the radial unit vector and f an arbitrary function. Let S be a sphere of radius R centered at the origin $r=0$. Using polar coordinates, $d\sigma = R^2\sin\theta d\theta d\phi$ and $\mathbf{n}\cdot\mathbf{e}_r = 1$, the integral of $\mathbf{v}$ over S evaluates to

$$\int_S d\sigma\, \mathbf{v}\cdot\mathbf{n} = f(R)R^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi} d\phi = 4\pi R^2 f(R).$$

On the other hand (cf. section **??** for a discussion of vector differential operators in curvelinear coordinate systems)

$$\nabla\cdot(f\mathbf{e}_r) = \nabla f\cdot\mathbf{e}_r + f\nabla\cdot\mathbf{e}_r = f' + 2f/r.$$

Evaluating the r.h.s. of Gauß' theorem

$$\int_{V(S)} d^3x\,\nabla\cdot\mathbf{v} = \int_0^R r^2dr\int_0^\pi d\theta\int_0^{2\pi} d\phi\,(f'(r)+2f(r)/r) = 4\pi\int_0^R drd_r(r^2f) = 4\pi R^2 f(R),$$

we obtain the same answer as above.

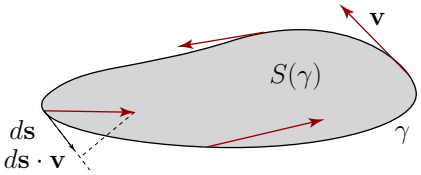


Figure 1.3: Schematic on Stokes law

Stokes law: Let γ be a smooth closed path in $\mathbb{R}^3$ and $S(\gamma)$ any surface bounded by γ. Further, let $\mathbf{v}$ be a vector field defined on all of S (including its boundary $\partial S = \gamma$.) Then,

$$\boxed{\oint_\gamma d\mathbf{s} \cdot \mathbf{v} = \int_{S(\gamma)} d\sigma \mathbf{n} \cdot (\nabla \times \mathbf{v}).} \tag{1.8}$$

▷ Example: Consider a vector $\mathbf{v}$ field with cylindrical symmetry, i.e. $\mathbf{v} = f(\rho)\mathbf{e}_\phi$, where cylindrical coordinates (ρ, ϕ, z) are implied. Integrating $\mathbf{v}$ over a circular contour γ of radius R around the symmetry axis $\rho = 0$ of the problem, we obtain (exercise)

$$\oint_\gamma d\mathbf{s} \cdot \mathbf{v} = R \int_0^{2\pi} d\phi f(R) = 2\pi R f(R).$$

To obtain the same result by Stokes theorem, we compute

$$\nabla \times \mathbf{v} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho f(\rho))\mathbf{e}_z.$$

Thus, choosing for $S(\gamma)$ a circular disc with radius R ($d\sigma = \rho d\rho\, d\phi$; $\mathbf{n} = \mathbf{e}_z$),

$$\int_{S(\gamma)} d\sigma\, \mathbf{n} \cdot (\nabla \times \mathbf{v}) = \int_0^{2\pi} d\phi \int_0^R \rho d\rho \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho f(\rho)) = 2\pi R f(R),$$

In agreement with the results of the line integration.

Notice that both Stokes and Gauß' law relate the integral of the derivative of a vector field over a certain manifold to the integral of the same vector field over the boundary of the manifold. The laws above are but two representatives of a family of identities of this type.[8] All these laws take the symbolic form

$$\int_M d\omega = \int_{\partial M} \omega, \tag{1.9}$$

where M is a manifold with boundary ∂M and 'd' stands for a derivative operation acting on some differentiable entity ω. (In fact, the notation above is not quite as symbolic as it may seem. It is shown in the theory of differential form that both (1.7) and (1.8) law derive from a general variant of Stokes theorem which takes exactly the form of (1.9).)

From Gauß's law one may derive a number of secondary integral identities which will be useful below: Applying (1.7) to a vector field of the form $\mathbf{v} = \theta\nabla\psi$, where θ and ψ are functions, and using the identity $\nabla \cdot (\theta\nabla\psi) = \nabla\theta \cdot \nabla\psi + \theta\Delta\psi$ we obtain **Green's first identity**

$$\int_S d\sigma\, \theta \partial_n \psi = \int_{V(S)} d^3r \left(\nabla\theta \cdot \nabla\psi + \theta\Delta\psi\right), \tag{1.10}$$

[8] ... the simplest example of which is the one–dimensional identity $\int_a^b dx \partial_x f = f(b) - f(a)$. (Think of the function f as a one–component vector field and the interval $[a, b]$ as an integration domain with 'boundary' $\partial[a, b] = \{a\} \cup \{b\}$.)

where $\partial_n\psi$ is the derivative of the function ψ in the direction normal to the surface S at the point of the surface element $d\sigma$.

Writing down Green's first identity with the role of θ and ψ interchanged and subtracting it from (1.10), we obtain **Green's second identity**

$$\int_S d\sigma \left(\theta\partial_n\psi - \partial_n\theta\psi\right) = \int_{V(S)} d^3r \left(\theta\Delta\psi - \psi\Delta\theta\right). \tag{1.11}$$

We begin with a substantiated discussion of the notion of **charge**. Empirically, we know that

- ▷ The existence of charge is tied to matter.
- ▷ charge is a scalar quantity that can be measured: Fixing a certain reference charge (for example 1 esu or 1C), we may assign a value to other charges by comparison[9] with that reference charge.
- ▷ The value of the charge $Q_V \leftrightarrow V$ assigned to a volume V of matter can be positive and negative.
- ▷ Charge is an 'extensive' scalar quantity which means that if two disjoint regions of space, V_1 and V_2, $V_1 \cap V_2 = \{\ \}$, contain charges Q_{V_1} and Q_{V_2}, respectively, then the total charge contained in $V_1 \cup V_2$ is given by $Q_{V_1\cup V_2} = Q_{V_1} + Q_{V_2}$.
- ▷ Charge is a 'conserved' quantity, i.e. as far as we know, nowhere in universe is charge created. (This, however, does not exclude the creation of positive and negative charges of equal and opposite magnitude.)

These features motivate the definition of a **charge density** distribution as follows: For a given point in space, $\mathbf{x}$, let $V_1 \supset V_2 \supset \ldots$ be a sequence of volume elements converging onto the point $\mathbf{x}$. Let Q_n be the charge contained in V_n.[10] We then define the charge density at $\mathbf{x}$ as

$$\rho(\mathbf{x}, t) \equiv \lim_{n\to\infty} \frac{Q_n(t)}{V_n}. \tag{1.12}$$

With this definition, the total charge contained in a finite region of space, V, is given by

$$Q_V(t) = \int_V d^3x\, \rho(\mathbf{x}, t). \tag{1.13}$$

Since charges cannot be spontaneously created, the only way for Q_V to change is that charges enter or leave the volume V through its surface $\partial V = S$. We define the **(charge) current** through S by

$$I_S(t) + \dot{Q}_{V(S)}(t) = 0, \tag{1.14}$$

i.e. the total current flowing through the surface S equals the rate at which the charge contained in V changes. This is an example of a **global conservation law**.

We wish to generalize the local definition of current to a refined, local one: Define a **current density** vector field $\mathbf{j}(\mathbf{r}, t)$ by demanding that for any oriented surface element $d\sigma$ with normal vector

[9] In practice, the notion 'comparison' implies an experiment, e.g., the measurement of the force acting on a charge.
[10] Notice that $Q_n = Q_n(t)$ may depend on time.

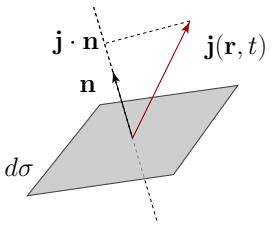
$\mathbf{n}(\mathbf{r})$, $d\sigma(\mathbf{j}\cdot\mathbf{n})dt$ is the amount of charge passing through $d\sigma$ in the time interval $[t, t+dt]$ (why is this a genuine definition?) The amount of charge $dQ_{S,\text{out}}$ flowing during the interval $[t, t+dt]$ through a surface S of finite extent obtains by integration $dQ_{S,\text{out}} = dt\int_S d\sigma\,(\mathbf{n}\cdot\mathbf{j})$. Defining the **current** through S, I_S by the rate of charge transfer, $I_S = dQ_{S,\text{out}}/dt$, we obtain

$$I_S(t) = \int_S d\sigma\,(\mathbf{j}\cdot\mathbf{n}).$$

Specifically, for a closed surface, $-d_tQ_{V(S)} = d_tQ_{S,\text{out}} = I_S$, as stated by the global conservation law above. Further, recalling the definition of charge Eq. (1.13) and the conservation law (1.14), we obtain the $\int_V d^3x\,(\nabla\cdot\mathbf{j} + \partial_t\rho) = 0$. This equation must hold for any choice of the reference volume V. The only way to satisfy this condition is to require the **continuity equation**

$$\boxed{\partial_t\rho + \nabla\cdot\mathbf{j} = 0.} \tag{1.15}$$

Eq. (1.15) represents the 'local formulation' of the conservation of current. Advantage of the global formulation: It is arguably more intuitive. Advantage of the local formulation: It does not make reference to a test volume and it is 'local', i.e. makes a statement about charges and currents at one given instance of space–time.



A shorthand notation: Define the — at this stage purely formal — notation: $\partial_\mu \equiv (\partial_t, \nabla)$, $\mu = 0,1,2,3$. Then the continuity equation assumes the form:

$$\partial_\mu j^\mu = 0, \tag{1.16}$$

where $j^{\mu=0} = \rho$, $j^{\mu=1,2,3} = j^{i=1,2,3}$ are the components of the **four–vector current density**.[11]

Notice that current conservation is implied by the Maxwell equations: Taking the divergence of $c\times$(1.2) and adding it to the time derivative of (1.1), we obtain

$$\underbrace{c\nabla\cdot(\nabla\times\mathbf{H})}_{0} = 4\pi(\partial_t\rho + \nabla\cdot\mathbf{j}).$$

Point charges and current loops

To facilitate the analysis of the Maxwell equations, we will often work with charge and current distributions that are 'singular' in that they are concentrated on infinitely sharply concentrated regions in space (mathematically: sets of measure zero.) Specifically, we will consider point–like charges (the analog of point particles in classical mechanics) and currents flows concentrated on mathematical curves (the latter representing theorists cartoons of 'infinitely thin wires'). To meaningfully represent such singular configurations in terms of the source functions ρ and $\mathbf{j}$, we need to introduce the notion of distributions.

▷ Math. How would one describe a 'point charge', i.e. an accumulation of a finite amount of charge, q, concentrated on a single point in space, in terms of a charge distribution ρ? What we need is a 'function' that is zero almost everywhere, except for one single point. Such functions have been popularized in physics (long before they made their way to mathematics) as δ–functions. Our objective here is to give the 'δ–function' a precise definition, and, equally important, to learn how to work with it.

[11] Following a widespread convention, we will denote the space like components of vectors by intermediate lowercase latin variables, $i, j, k, \ldots = 1,2,3$. (3+1)–dimensional space–time vectors are indexed by intermediate lowercase Greek variables, $\lambda, \mu, \nu, \ldots = 0,1,2,3$.

To start with, consider the case of functions $f : \mathbb{R} \to \mathbb{R}$ in one–dimensional space. Define: $\mathcal{D}(\mathbb{R}) = \mathcal{C}_c^\infty(\mathbb{R})$ where $\mathcal{C}_c^n(\mathbb{R})$ is the space of n–fold differentiable functions with compact support. (Elements of $\mathcal{D}$ are very benign. For reasons to be come clear momentarily, they are sometimes called **test–functions**.) Notice that $\mathcal{D}$ forms an (infinite–dimensional) real vector space: For two functions $\phi_1, \phi_2 \in \mathcal{D}$ and $c_1, c_2 \in \mathbb{R}$, $c_1\phi_1 + c_2\phi_2 \in \mathcal{D}$.[12]

A continuous functional on $\mathcal{D}$ is a continuous mapping $F : \mathcal{D} \to \mathbb{R}, \phi \mapsto F[\phi]$ assigning to every $\phi \in \mathbb{R}$ a number. A **continuous linear fuctional** is a functional that is linear: $F[c_1\phi_1+c_2\phi_2] = c_1F[\phi_1]+c_2F[\phi_2]$. A **distribution** F on $\mathbb{R}$ is a continuous linear functional subject to one additional condition: if the sequence of test functions $\phi_k \overset{k\to\infty}{\longrightarrow} \phi$ converges to the test functions ϕ, then $\lim_{k\to\infty} F[\phi_k] \to F[\phi]$.

▷ Example: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define the functional

$$F_f[\phi] \equiv \int_{-\infty}^{\infty} dx\, f(x)\phi(x). \tag{1.17}$$

(Why does F_f meet the defining criteria above?) Distributions of this type are called **regular distributions**. In a way, the function ϕ probes the properties of f, hence the denotation 'test function'.

Not every distribution is regular. As an important example we mention:

▷ Example: The ***δ*–distribution**: For $a \in \mathbb{R}$, define

$$\delta_a[\phi] = \phi(a). \tag{1.18}$$

The δ–distribution meets all the criteria of the definition above. Evidently, however, there is no regular function such that $\delta_a[\phi] = \int dx\, f(x)\phi(x)$.

Although the δ–distribution as such is not regular, one may prove the following statement: every distribution F can be represented as the limit $F = \lim_{n\to\infty} F_{f_n}$ of a sequence of regular distributions. (As we shall see, the functions f_n 'modeling' a irregular distribution are of considerable applied relevance.) Specifically, let us construct a sequence of functions f_n^a such that $F_{f_n^a} \to \delta_a$ converges to the δ–distribution:

$$f_n^a(x) = \frac{n}{\epsilon} \exp\left(-\frac{\pi n^2(x-a)^2}{\epsilon^2}\right).$$

The functions f_n^a are unit–normalized, $\int dx\, f_n^a(x) = 1$ and, for large enough n, sharply peaked around $x = a$. Taylor expanding the test–function ϕ around $x = a$, one verifies that

$$F_{f_n^a}[\phi] = \frac{n}{\epsilon} \int dx\, \exp\left(-\frac{\pi n^2(x-a)^2}{\epsilon^2}\right) \phi(x) = \phi(a) + \mathcal{O}(\epsilon^2/n^2),$$

i.e. $\lim_{n\to\infty} f_n^a[\phi] = \delta_a[\phi]$. In the physics community, it is customary to work with the (symbolic, because the limit does not exist) notation)

$$\delta(x-a) \equiv \lim_{n\to\infty} f_n^a(x), \tag{1.19}$$

and to think of $\delta(x-a)$ as a 'function', the ***δ*–function**, infinitely sharply peaked around a. Using this notation,

$$\delta_a[\phi] = \phi(a) = \int dx\, \delta(x-a)\phi(x) \equiv F_{\delta(x-a)}[\phi].$$

[12] More generally, the set $\mathcal{C}^n(U, \mathbb{R}^r)$ containing the n–fold differentiable functions $f : U \subset \mathbb{R}^d \to \mathbb{R}^r$ forms a vector space, too. (Functions of this type can be added and multiplied by numbers.) This is why we often speak of a *space* of functions of a certain type. The interpretation of functions as elements of (infinite dimensional) spaces will play a pivotal role below.

Remarks: (a) Of course, the δ–distribution may be generated by functions f_n different from the Gaussian functions above. (Invent examples.) (b) We will use the sloppy physics notation throughout.

For a differentiable function, f, a straightforward integration by parts shows that $F_{f'}[\phi] = -F_f[\phi']$. Applying this result to the sequence of functions converging onto the δ–distribution, we obtain a formal definition of the 'derivative of the δ–function',

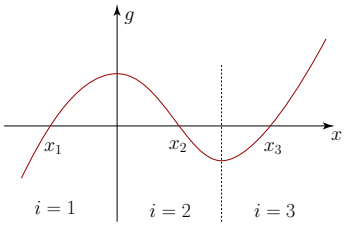
$$F_{\delta'(x-a)} = -F_{\delta(x-a)}[\phi'] = -\phi'(a).$$

Conversely, the δ–function may be obtained as the derivative of the Heaviside step function Θ. Of course, the step function $\Theta(x-a)$ is not differentiable in the conventional sense. (Its difference quotient is zero everywhere, except for $x = a$ where it diverges. Thus, the formal limit of the difference quotient, the 'derivative' has properties akin to a δ–function.). Formally, however, we may define

$$F_{\Theta'(x-a)}[\phi] = -F_{\Theta(x-a)}[\phi'] = -\int_{-\infty}^{\infty} dx\,\Theta(x-a)\phi'(x) = -\int_a^{\infty} dx\,\phi'(x) = \phi(a) = F_{\delta(x-a)}[\phi],$$

where in the second last equality we used that $\phi(\infty) = 0$ (compact support of test functions.) Since this equality holds for arbitrary functions testing the step function, we define $\frac{d}{dx}\Theta(x-a) = \delta(x-a)$.

For an arbitrary differentiable function g and a sequence of functions $f_n(x)$ converging onto $\delta(x)$, we define $\delta(g(x)) \equiv \lim_{n\to 0} f_n(g(x))$. Evaluating the test function integral,



$$F_{\delta(g)}[\phi] = \int dx\,\delta(g(x))\phi(x) = \sum_i \int \frac{dg}{|g'(x)|}\,\delta(g)\phi(g^{-1}(x)) =$$
$$= \sum_i \frac{1}{|g'(x_i)|}\phi(g^{-1}(x_i)),$$

where the sum extends over all those regions where g is invertible (see the figure.) and x_i are the zeros of g in these domains. In the notation of the δ–function:

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x-x_i). \tag{1.20}$$

To summarize our so far results (and a few more that are trivially proven):

- ▷ $\delta(x-a)f(x) = \delta(x-a)f(a)$,
- ▷ $\delta(x) = \delta(-x)$,
- ▷ $\Theta'(x-a) = \delta(x-a)$,
- ▷ $\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x-x_i)$,
- ▷ $\delta(cx) = |c|^{-1}\delta(x)$.

The **generalization to higher dimensions** is straightforward: Assuming $d = 3$ for definiteness, we define

$$\delta(\mathbf{x}-\mathbf{a}) = \prod_{i=1}^{3}\delta(x_i - a_i).$$

With this definition $\int d^3x\, f(\mathbf{x})\delta(\mathbf{x}-\mathbf{a}) = f(\mathbf{a})$. The generalization of Eq. (1.20) becomes (exercise)

$$\delta(\mathbf{g}(\mathbf{x})) = \sum_i \left|\frac{\partial \mathbf{g}}{\partial \mathbf{x}_i}\right|^{-1} \delta(\mathbf{x}-\mathbf{x}_i),$$

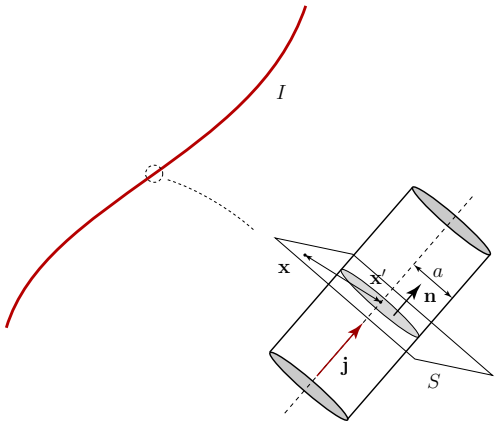
where $\mathbf{g}:\mathbb{R}^3\to\mathbb{R}^3$ is a function and $|\partial\mathbf{g}/\partial\mathbf{x}|$ a shorthand notation for the determinant of the Jacobi matrix. This equation can be used, e.g., to compute the form of the δ–function in non–cartesian coordinates. For example, in polar coordinates,

$$\delta(\mathbf{x}) = \frac{1}{r^2\sin\theta}\delta(r)\delta(\theta)\delta(\phi).$$

(Check that this function obeys the defining properties of the δ–function.)

As with the 'point particles' of classical mechanics, it is often useful to consider **point charges**, i.e. infinitely small regions $V \searrow 0$ in space centered around a reference point $\mathbf{x}$ and carrying a fixed amount of charge $Q_{V\searrow 0} = q$. For example, we may chose $V(a)$ to be a ball of radius a carrying a charge density $\rho_a(\mathbf{x}) = \frac{3q}{4\pi a^3}1_V(\mathbf{x})$, where $1_V(\mathbf{x})$ is the characteristic function of V (unity if $\mathbf{x}\in V$, zero otherwise.) Evidently, $Q_{V(a)} = q$, i.e. the shrinkage of the volume for $a\searrow 0$ is compensated by a diverging charge density. At the same time, the functions ρ_a converge onto a three–dimensional representation of the δ–function, i.e.

$$\rho(\mathbf{x}) = \lim_{a\searrow 0}\rho_a(\mathbf{x}) \equiv q\delta(\mathbf{x}-\mathbf{x}_0), \tag{1.21}$$



The current–analog of a point charge is a **current loops**, i.e. a curve in space carrying a uniform current I. To obtain an operational definition of a current loop, consider a curve γ in space. For most (all modulo sets of measure zero) points $\mathbf{x}$ in space, we may identify a point $\mathbf{x}'\in\gamma$ that is closest in to $\mathbf{x}$. (Just minimize the distance function $|\mathbf{x}-\mathbf{x}'(t)|$, where $\mathbf{x}'(t)$ is an explicit parameterization of γ.) Define $\mathbf{j}_a(\mathbf{x}) = \mathbf{n}_{\mathbf{x}'}(I/\pi a^2)\Theta(a-|\mathbf{x}-\mathbf{x}'|)$, where $\mathbf{n}_{\mathbf{x}'} = d_t\mathbf{x}'(t)/|\mathbf{x}'(t)|$ is the unit vector along γ at $\mathbf{x}'$ (cf. the figure.) To see that this definition does the job, compute the current through any surface S cutting through γ. Without loss of generality, we may assume that $\mathbf{S}$ is planar and perpendicular to γ (why does this follow from charge conservation?). Then, $\mathbf{n}_{\mathbf{x}'} = \mathbf{n}$ equals the vector normal to S and $I = \int_S d\sigma\,\mathbf{j}\cdot\mathbf{n} = 2\pi\int_0^a \rho d\rho(I/\pi a^2) = I$, independent of a we define the current density of a loop as $\mathbf{j} = \lim_{a\searrow 0}\mathbf{j}_a$.

We next show that an explicit representation of the limiting distribution is given by

$$\mathbf{j}(\mathbf{x}) = I\int dt\,\dot{\mathbf{x}}(t)\delta(\mathbf{x}(t)-\mathbf{x}). \tag{1.22}$$

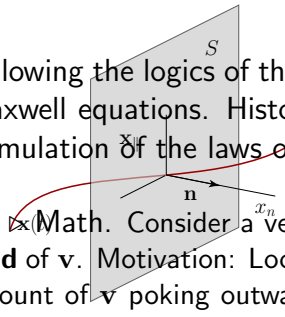
To prove this assertion, let us again demonstrate that the total current through any plane S cutting through the curve is given by I. Choosing coordinates so that at time t_0 the curve pierces

S at $\mathbf{x} = 0$ and denoting the unit vector normal to $\mathbf{S}$ by $\mathbf{n}$, we obtain

$$I_S = \int_S d\sigma\, \mathbf{n}\cdot\mathbf{j} = I \int dt \int d\sigma\, \dot{x}_n(t)\delta(\mathbf{x}_\parallel(t))\delta(x_n(t)) =$$
$$= I \int dt \int d\sigma\, \dot{x}_n(t)\delta(\mathbf{x}_\parallel - \mathbf{x}_\parallel(t))\frac{\delta(t-t_0)}{d_t x_n(t)} = I\int d\sigma\, \delta(\mathbf{x}_\parallel(t_0)) = I,$$

where $\mathbf{x}_\parallel$ is the two–dimensional in–plane component of $\mathbf{x}$ and we used (1.20).

1.2.5 Integral form of the Maxwell equations



Following the logics of the previous section in reverse order, we derive 'global' representations of the Maxwell equations. Historically, the – experimental — discovery of these global laws preceded the formulation of the laws of ED as differential equations by Maxwell.

Math. Consider a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$. We denote the scalar field $f(\mathbf{x}) = \nabla\cdot\mathbf{v}$ as the **source field** of $\mathbf{v}$. Motivation: Loosely speaking, the integral of $\mathbf{v}$ over any closed surface is determined by the net amount of $\mathbf{v}$ poking outwardly through that surface. At the same time, this integral (Gauß' law) equals the integral of f over the volume enclosed by the surface. Imagining outwardly directed field lines[13] as emanating from some 'sources', we identify $f(\mathbf{x})dV$ as the source–contents of the volume element dV.

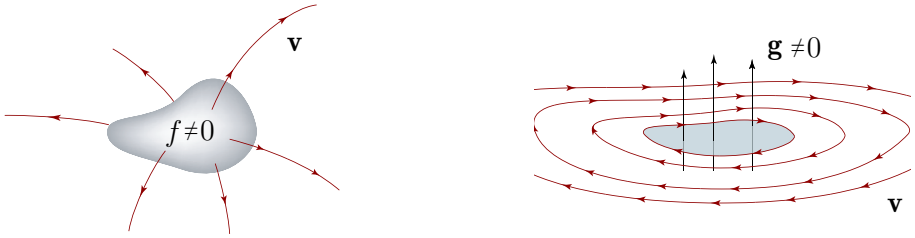


Figure 1.4: Schematic on the source and the circulation contents of vector fields

Similarly, the vector field $\mathbf{g} \equiv \nabla\times\mathbf{v}$ is called the **circulation field** of $\mathbf{v}$. The field $\mathbf{v}$ contains circulating contributions if, for some closed contours γ, $\oint_\gamma d\mathbf{s}\cdot\mathbf{v} \neq 0$. Since (Stokes) $\oint_\gamma d\mathbf{s}\cdot\mathbf{v} = \int_{S(\gamma)} d\sigma\,\mathbf{n}\cdot\mathbf{g}$, a non–vanishing of $\mathbf{g}$ implies that curves with this property exist (chose γ to be a small curve winding around a region in space where $\mathbf{g}$ is non–vanishing and of definite sign.)

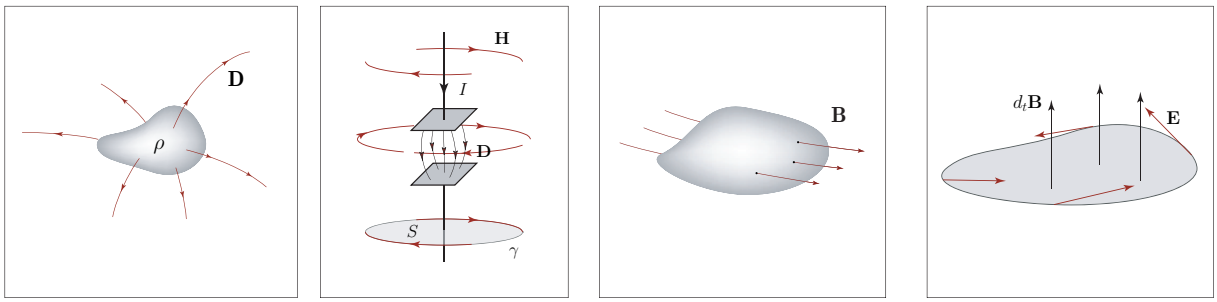
Gauß's law (physics)

According to experimental observation,[14] electric charges are the sources of the electric (displacement) field:

$$\int_S d\sigma\, \mathbf{D}\cdot\mathbf{n} = 4\pi \int_{V(S)} d^3x\, \rho = 4\pi Q_{V(S)}, \tag{1.23}$$

[13] Loosely speaking, the **field lines** of a vector field are curves in space locally tangent to the field (i.e. a visualization of the field is obtained by attaching arrows tangent to the field lines.) The more precise definition goes as follows: For each point $\mathbf{x}$ in space define a curve $\Phi_\mathbf{x}(t)$ such that $d_t\Phi_\mathbf{x}(t)\big|_{t=0} = \mathbf{v}(\mathbf{x})$. $\Phi_\mathbf{x}(t)$ traces out the field line through $\mathbf{x}$. The map $\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3, (\mathbf{x},t)\mapsto\Phi_\mathbf{x}(t)$ is called the **flux** of the vector field.

[14] Experimentally, the field $\mathbf{D}$ is determined by measuring the force $\mathbf{F} = \epsilon_0^{-1} q\mathbf{D}$ locally acting on a test charge q.



a) b) c) d)

Figure 1.5: Schematic on the integral form of Maxwells equations. a) Gauß's law, b) circulation of the magnetic field, c) absence of magnetic charges, d) law of induction

where S is an arbitrary closed surface, i.e. the amount of field lines leaving the volume $V(S)$ is determined by the charge $Q_{V(S)}$ inside $V(S)$ (cf. Fig. 1.5 a).) Eq. (1.23) is what physicists denote as **Gauß's**. The qualifying attribute '(physics)' is appropriate because the empirical law (1.23) and the mathematical law (1.7) need to be distinguished.

Applying (1.7) to the l.h.s. of this equation and using the arbitrariness of S (cf. our argumentation in the previous section), we obtain the Maxwell equation (1.1).

Circulation of the magnetic field

Consider a surface S in space pierced by electric current. Experimentally, one finds that the line integral of the magnetic[15] field circulating around that surface equals that current. Likewise, a surface pierced by non–vanishing temporally fluctuating displacement field also supports a magnetic field circulation (cf. Fig. 1.5 b).) These observations are summarized in the law

$$\int_{S(\gamma)} d\sigma\,(\mathbf{j} + d_t\mathbf{D}) \cdot \mathbf{n} = \int_{\gamma} d\mathbf{s} \cdot \mathbf{H}. \tag{1.24}$$

Application of Stokes law to the r.h.s. obtains $\forall S: \int_S d\sigma\,(\nabla \times \mathbf{H} - d_t\mathbf{D} - \mathbf{j}) \cdot \mathbf{n} = 0$, i.e. the Maxwell equation (1.2).

Absence of magnetic charges

To the best of our knowledge no isolated magnetic charges exist in the universe; the net flux[16] of the magnetic field through any closed surface equals zero: $\forall S: \int_S d\sigma \mathbf{B} \cdot \mathbf{n} = 0$ (cf. Fig. 1.5 c).) Application of Gauß' law to this equation yields the Maxwell equation (1.4).

[15] The magnetic field can be determined, e.g., by measuring the torque exerted on a current loop.

[16] It is customary to denote the integral $\Phi_S(\mathbf{v}) = \int d\sigma\,\mathbf{n} \cdot \mathbf{v}$ of a vector field over a surface S as the **flux** of the field through S. Do not confuse this quantity with the flux Φ introduced previously!

Law of induction

Consider a time dependent magnetic induction $\mathbf{B}$ piercing a surface S bounded by a curve γ. One observes that the circulation of the electric field strength around γ is determined by the time derivative of the flux of $\mathbf{B}$ through S (cf. Fig. 1.5 d)):

$$\oint_\gamma d\mathbf{s} \cdot \mathbf{E} = -\frac{1}{c} \int_S d\sigma\, \mathbf{n} \cdot d_t \mathbf{B}.$$

Applying Stoke's theorem to the l.h.s. of this equation, we find $\forall S: \int\int_S d\sigma\, \mathbf{n} \cdot (\mathbf{E} + c^{-1} d_t \mathbf{B}) = 0$, equivalent to the Maxwell equation (1.3)

Chapter 2

The static limit

So far: discussed mathematical structure of Maxwell equations and outlined connection to experimentally observable phenomena. Now: develop solution theory of Maxwell equations.

2.1 Decoupling of the Maxwell equations in the static limit

Main source of complexity: coupling of the four equations to an in–separable entity. Drastic simplifications arise in cases where the fields are static: $\partial_t \mathbf{E} = \ldots = \partial_t \mathbf{H} = 0$. What are necessary and sufficient conditions for time–independence of the fields?

Equation (1.1) implies that $\nabla \cdot \partial_t \mathbf{D} = 4\pi \partial_t \rho$, i.e. $\partial_t \mathbf{D} = 0$ necessarily implies $\partial_t \rho = 0$, the time–independence of all charge distributions. This in turn enforces (Eq. (1.15)) that $\nabla \cdot \mathbf{j} = 0$, i.e. in the static limit there are no sources of electric currents, all field lines of the current field are closed. Differentiating the Maxwell equation (1.2) w.r.t. time, and noting that $\partial_t \mathbf{H} = \partial_t \mathbf{D} = 0$ we further conclude that the current flow is stationary, $\partial_t \mathbf{j} = 0$. To summarize, we find that

> The time–independence of the EM fields necessarily implies the stationarity of charge densities, $\partial_t \rho = 0$, and of the currents $\partial_t \mathbf{j} = 0$; in the static limit, the current–field is source–free, $\nabla \cdot \mathbf{j} = 0$.

Arguing in reverse, we consider the Maxwell equations for stationary sources $\partial_t \rho = 0$ and $\partial_t \mathbf{j} = 0$. In the following we will show that under these conditions stationary solutions $\mathbf{E}, \cdot, \mathbf{H}$ can be found. Adopting stationarity of the fields as a working hypothesis, we obtain the **static limit of the Maxwell equations**

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho, \\ \nabla \times \mathbf{E} &= 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \nabla \times \mathbf{H} &= \frac{4\pi}{c}\mathbf{j}, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{2.2}$$

Importantly, the set of equations decomposes into two groups, one describing the 'electric sector' of the theory, with the charge density as a source, the other the magnetic sector, with the current density as a source.

2.2 Electrostatics

2.2.1 Formulation of the problem

We begin by exploring the theory of static electric fields, as described by the basic equations (2.1) (plus the vacuum relation $\mathbf{D} = \epsilon_0 \mathbf{E}$.) Exploiting that we are working in the Gauß system, we will set $\epsilon_0 = 1$ throughout and indicate the ϵ_0–dependence of the theory only in final formulae.

The fact that $\mathbf{E}$ is circulation–less, $\nabla \times \mathbf{E}$, implies[1] that $\mathbf{E}$ can be written as the gradient of a scalar function ϕ:

$$\boxed{\mathbf{E} \equiv -\nabla\phi}, \tag{2.3}$$

where $\phi : \mathbb{R}^3 \to \mathbb{R}, \mathbf{x} \to \phi(\mathbf{x})$ is the **scalar potential** of the theory. Substituting this ansatz into the first of Eqs.(2.1), we obtain the **Poisson equation**

$$\boxed{\Delta\phi = -4\pi\rho.} \tag{2.4}$$

Once this equation is solved, the problem is under control for a (Eq. (2.3)) will yield the electric field strength $\mathbf{E}$.

▷ Info. For a given point in space, $\mathbf{x}_0$, the equation $\phi(\mathbf{x}) = \phi(\mathbf{x}_0)$ defines a two–dimensional surface in the vicinity of $\mathbf{x}_0$ – an **equipotential surface**. The electric field $\mathbf{E}(\mathbf{x}_0)$ points in the direction of the steepest descent of ϕ at $\mathbf{x}$. i.e. it is perpendicular to the equipotential surfaces of the problem. Once the potential is known we may, thus, easily visualize the electric field.

Thus, our central objective will be to develop solution strategies for the Poisson equation. For later reference we note that in the case of a vanishing charge distribution, $\rho = 0$, the Poisson equation is denoted as **Laplace equation**

$$\Delta\phi = 0. \tag{2.5}$$

▷ Math. As a first useful application of the δ–function, we introduce the concept of **Green functions**. Suppose we want to solve a linear differential equation $Lf = g$, where $f, g \in \mathcal{C}^n(U \subset \mathbb{R}^d, \mathbb{R})$ are functions (f is the solution we are after) and L is a **linear differential operator**, i.e. a an expression of the type

$$L = h^{(0)}(\mathbf{x}) + \sum_i h_i^{(1)}(\mathbf{x})\partial_{x_i} + \sum_{ij} h_{ij}^{(2)}(\mathbf{x})\partial^2_{x_i x_j}, \tag{2.6}$$

[1] Proof: For a given point in space $\mathbf{r}$, an arbitrary fixed reference point $\mathbf{r}_0$, and an equally arbitrary curve γ connecting $\mathbf{r}_0$ and $\mathbf{r}$, define $\phi_\gamma(\mathbf{r}) \equiv -\int_\gamma d\mathbf{s} \cdot \mathbf{E}$. Then $\nabla\phi(\mathbf{r}) = -\mathbf{E}(\mathbf{r})$. To see that $\phi_\gamma \equiv \phi$ is independent of the choice of γ, compute $\phi_\gamma - \phi_{\gamma'} = \oint_{\gamma-\gamma'} d\mathbf{s} \cdot \mathbf{E} = \int_{S(\gamma-\gamma')} d\sigma\, \mathbf{n} \cdot \nabla \times \mathbf{E} = 0$, where $\gamma - \gamma'$ is the closed curve obtained by concatenating the reverse of γ' with γ.

where the h's are some functions and the derivatives act on the argument function f. Notice that $L(c_1 f_1 + c_2 f_2) = c_1 L f_1 + c_2 L f_2$, i.e. L acts linearly in function space (hence the denotation *linear* differential operator.) Many problems in electrodynamics (and for that matter quantum mechanics) effectively reduce to the solution of linear partial differential equation of the type above. For example, the Poisson equation (2.4) belongs to this category ($L \leftrightarrow -\Delta$, $f \leftrightarrow \phi$, $g \leftrightarrow 4\pi\rho$.)

The weak spot of a linear differential equation is, indeed, its linearity. If f_1 and f_2 solve the equation for g_1 and g_2, respectively, the function $f_1 + f_2$ will be a solution for the inhomogeneity $g_1 + g_2$. This feature suggests to first solve the equation for functions g which are as simple as possible. Arguably the simplest function at all is the δ–function: except for one point in space it vanishes everywhere, i.e. it is maximally structureless. We may employ the δ–function to represent an arbitrary inhomogeneity as

$$g(\mathbf{x}) = \int d^d x' \, \delta(\mathbf{x} - \mathbf{x}') g(\mathbf{x}'),$$

i.e. as a sum (integral) over point–localized inhomogeneities centered at $\mathbf{x}'$ of unit–normalization ($\int \delta = 1$) each carrying a weight $g(\mathbf{x}')$. It is, thus, a good idea to first solve the equation

$$L G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \tag{2.7}$$

where the derivatives in L act only on the argument $\mathbf{x}$. The function — a distribution, in fact — G is known as the **Green function** of the differential equation. Once the Green function is known, the general solution of the differential equation can be obtained as by summation (integration) over the prototype solutions G:

$$f(\mathbf{x}) = \int d^3 x' \, G(\mathbf{x}, \mathbf{x}') g(\mathbf{x}').$$

Indeed, by direct substitution, $L f(\mathbf{x}) = \int d^3\mathbf{x}' \, L G(\mathbf{x}, \mathbf{x}') g(\mathbf{x}') = \int d^3 x' \, \delta(\mathbf{x} - \mathbf{x}') g(\mathbf{x}') = g(\mathbf{x})$. The linearity of the problem thus enables us to break down the solution of the general problem into the (generally much simpler) solution of the point–inhomogeneity problem, plus a subsequent integration.

Now, one may ask whether Eq. (2.7) does have a unique solution at all. Referring for a substantiated discussion to below, we here merely note that the solution becomes unique once appropriate boundary conditions for $G(\mathbf{x}, \mathbf{x}')|_{\mathbf{x} \to \partial U}$ at the boundaries of the domain of definition are specified. For unspecified boundary conditions, the solution is not unique: For any G, the function $G + F$ will be a solution as well, provided F solves the homogeneous problem $L F(\mathbf{x}) = 0$. Differential equations whose solution requires the specification of data at the boundaries of the domain of support are known as **boundary value problems**.

2.2.2 Boundary value problem I: infinite space

Potential created by a static charge distribtution

We next specialize to the case of the Poisson equation. Using the notation of the previous section, $L = -\Delta$, $f = \phi$ and $g = 4\pi\rho$. In this section, we will consider the Poisson equation in infinite three–dimensional case, i.e. the domain of definition of the differential equation is $U = \mathbb{R}^3$. As for the boundary conditions to be imposed on Φ, we require — on physical grounds — that the electric field $\mathbf{E}$ created by any spatially localized charge distribution vanish at infinity (for otherwise infinitely separated charges would exert a force on each other.) This condition is satisfied if $\phi(\mathbf{r} \to \infty) \to 0$,

i.e. ϕ vanishes at the boundaries of U, a ball of infinite radius. (We will see below, that this condition is, in fact, necessary to ensure the vanishing of the electric field.)

According to the logics developed in the previous problem, the solution of the problem is (up to an integration) equivalent to the computation of the **Green function of electrostatics**,

$$\Delta G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'). \tag{2.8}$$

Physically, $G(\mathbf{x},\mathbf{x}')$ is proportional to the potential of a point charge at $\mathbf{x}'$ felt at $\mathbf{x}$. Notice that this potential can depend only on the difference vector, $\mathbf{x}-\mathbf{x}'$, i.e. $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}-\mathbf{x}')$.

We next discuss two strategies for solving Eq.(2.8):

Solution strategy I: (Physics inspired but very restrictive.) Without loss of generality, we chose coordinates such that $\mathbf{x}' = 0$. Physically, we expect $G(\mathbf{x}) = G(r)$ to depend only on the distance to the origin.[2] To determine the r–dependence of $G(r)$ let us integrate the vector field $\nabla G(r) = \partial_r G(r)\mathbf{e}_r$ (which is proportional to the electric field created by the point charge at the origin, why?) over a sphere S of radius R centered at the origin $\mathbf{x} = 0$ (cf. the example on p **??**: $\int_S d\sigma\, \mathbf{n}\cdot\nabla G = 4\pi R^2 \partial_R G(R) = \int_{V(S)} d^3x \underbrace{\nabla\cdot\nabla}_{\Delta} G = -\int_{V(S)} \delta(\mathbf{x}) = -1$. Solving for G we obtain $\partial_R G(R) = -1/4\pi R^2$ which integrates to $G(R) = 1/4\pi R$. Switching back to vectorial coordinates we obtain the important result

$$\boxed{G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}.} \tag{2.9}$$

Before discussing this any further, let us discuss the much more sophisticated

▷ Math. Recall that the **Fourier transform** of a function $f \in C^n(\mathbb{R}^d,\mathbb{R})$ is defined by

$$\boxed{\tilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^d}\int d^dx\, f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}},} \tag{2.10}$$

(provided the integral exists.) Note: to enlarge the class of functions that can be Fourier transformed, it is customary to generalize the definition above according to

$$\tilde{f}(\mathbf{k}) = \lim_{\epsilon\to 0}\frac{1}{(2\pi)^d}\int d^dx\, f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}-\epsilon\mathbf{x}^2}.$$

Remarks: (a) every function $f(\mathbf{x})$ that grows sub–exponentially for large $|\mathbf{x}|$ can be transformed. (b) It is customary to suppress the convergence generating factor ϵ if it is not needed. (c) For notational simplicity, we will often denote the Fourier transform by f instead of $\tilde{f}$.

The function $f(\mathbf{x})$ is retrieved by the inverse transformation[3]

$$f(\mathbf{x}) = \int d^dk\, e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{f}(\mathbf{k}).$$

[2] More formally, this feature follows from the fact that both $\delta(\mathbf{x})$ and the Laplace operator Δ are invariant under rotations in space. Thus, the solution to $\Delta G(\mathbf{x}) = \delta(\mathbf{x})$ will be rotationally invariant, too.

[3] As with the $\mathbf{x}\to\mathbf{k}$ direction of the transformation, the inverse transform should be understood as

$$f(\mathbf{x}) = \lim_{\epsilon\to 0}\int d^dk\, e^{i\mathbf{k}\cdot\mathbf{x}-\epsilon\mathbf{k}^2}\tilde{f}(\mathbf{k}).$$

▷ Example: The function $f(\mathbf{x}) = 1$ can not be transformed in the restrictive sense (2.10). However

$$\tilde{f}(\mathbf{k}) = \lim_{\epsilon\to 0} \frac{1}{(2\pi)^d} \int d^d x\, f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x} - \epsilon\mathbf{x}^2} = \lim_{\epsilon\to 0} \frac{1}{(4\pi\epsilon)^{d/2}} e^{-\mathbf{k}^2/4\epsilon} = \delta(\mathbf{k}),$$

where the last identity follows by comparison with (the d–dimensional generalization of) (1.19). Indeed, for a general function the substitution of the inverse Fourier transform into (2.10) demands $f(\mathbf{k}) \stackrel{!}{=} \frac{1}{(2\pi)^d} \int d^d x \int d^d k\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} f(\mathbf{k}')$ which implies the important identity

$$\boxed{\frac{1}{(2\pi)^d} \int d^d x\, e^{i\mathbf{k}\cdot\mathbf{x}} = \delta(\mathbf{k}).} \tag{2.11}$$

Reminder of the most important features of the Fourier transform. (The proofs follow readily from Eq. (2.10).):

function	trafo.	function	trafo.
$f(\mathbf{x})$	$\tilde{f}(\mathbf{k})$	$f^*(\mathbf{x})$	$\tilde{f}^*(-\mathbf{k})$
$f(\mathbf{x}) + g(\mathbf{x})$	$\tilde{f}(\mathbf{k}) + \tilde{g}(\mathbf{k})$	$af(\mathbf{x})$	$a\tilde{f}(\mathbf{k})$
$\partial_{x_j} f(\mathbf{x})$	$ik_j \tilde{f}(\mathbf{k})$	$-ix_j f(\mathbf{x})$	$\partial_{k_j} \tilde{f}(\mathbf{k})$
1	$\delta(\mathbf{x})$	$\delta(\mathbf{x})$	$(2\pi)^{-d}$
$f(\mathbf{x}+\mathbf{a})$	$e^{-i\mathbf{k}\cdot\mathbf{a}} \tilde{f}(\mathbf{k})$	$e^{-i\mathbf{x}\cdot\mathbf{a}} f(\mathbf{x})$	$\tilde{f}(\mathbf{k}+\mathbf{a})$
$f(a\mathbf{x})$	$\lvert a\rvert^{-d} \tilde{f}(\mathbf{k}/a)$	$(f * g)(\mathbf{x})$	$(2\pi)^d \tilde{f}(\mathbf{k})\tilde{g}(\mathbf{k})$

Table 2.1: Important features of the Fourier transform

In the last line, the **convolution** of two functions is defined as

$$(f * g)(\mathbf{x}) = \int d^d x' f(\mathbf{x} - \mathbf{x}') g(\mathbf{x}').$$

Solution strategy II: (more laborious but highly generalizable) Consider a general linear partial differential equation $Lf = g$ with *constant coefficients*, i.e. one where the differential operator L (cf. Eq. (2.6)) contains only constant functions $h = \text{const.}$. (Our Poisson equation belongs to this family.) Differential equations of this type can readily be solved[4] by Fourier transformation:

Consider the equation (summation convention)

$$\left(h^{(0)} + h_i^{(1)} \partial_{x_i} + h_{ij}^{(2)} \partial^2_{x_i x_j} + \ldots\right) f(\mathbf{x}) = g(\mathbf{x}).$$

Fourier transforming both sides of this equation we obtain $(h^{(0)} + ih_i^{(1)} k_i - h_{ij}^{(2)} k_i k_j + \ldots)\tilde{f}(\mathbf{k}) = \tilde{g}(\mathbf{k})$ which is solved by $\tilde{f}(\mathbf{k}) = \tilde{g}(\mathbf{k})/(h^{(0)} + ih_i^{(1)} k_i - h_{ij}^{(2)} k_i k_j + \ldots)$. A solution of the equation is thus

[4] In the jargon of differential equations, 'solution' means transformation to a form that may still involve an integral operation.

given by

$$f(\mathbf{x}) = \int d^d k\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\tilde{g}(\mathbf{k})}{h^{(0)} + i h_i^{(1)} k_i - h_{ij}^{(2)} k_i k_j + \ldots},$$

i.e. the problem has been reduced to the computation of an integral. We write 'a' solution instead of 'the' solution because (for unspecified boundary conditions) to any f one may add a solution f_0 of the homogeneous equation $Lf_0 = 0$. Fourier transforming this latter equation, we obtain $(h^{(0)} + i h_i^{(1)} k_i - h_{ij}^{(2)} k_i k_j + \ldots)\tilde{f}_0(\mathbf{k}) = 0$ which has the solution (think why)

$$\tilde{f}_0(\mathbf{k}) = h(\mathbf{k})\, \delta(h^{(0)} + i h_i^{(1)} k_i - h_{ij}^{(2)} k_i k_j + \ldots).$$

We illustrate the general procedure on the example of the Poisson equation (2.8). Fourier transforming the equation, and setting $\mathbf{x}' = 0$ as before, we obtain the

$$\tilde{G}(\mathbf{k}) = \frac{1}{(2\pi)^3 k^2}.$$

The next and final task is to compute the inverse transform of this expression:

$$G(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3k\, \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2} = \frac{1}{(2\pi)^2} \int_0^\pi \underbrace{\sin\theta d\theta}_{-d(\cos\theta)} \int_0^\infty k^2 dk \frac{e^{ikr\cos\theta}}{k^2} =$$
$$= \frac{1}{2\pi^2} \int_0^\infty dk\, \frac{\sin kr}{kr} = \frac{1}{2\pi^2 r} \int_0^\infty dk\, \frac{\sin k}{k} = \frac{1}{4\pi r},$$

where in the second equality we introduced polar coordinates in $\mathbf{k}$–space, $\mathbf{k} \leftrightarrow (k, \phi, \theta)$ (with the vector $\mathbf{x}$ lying on the axis $\theta = 0$) and in the final equality used that $\int_0^\infty dk\, \frac{\sin k}{k} = \pi/2$. Generalizing to arbitrary values of the reference vector $\mathbf{x}'$, we obtain $G(\mathbf{x} - \mathbf{x}') = 1/4\pi|\mathbf{x} - \mathbf{x}'|$ in agreement with our earlier result (2.9).

According to the general scheme developed on p xxiv, the potential created by a general charge distribution is given by $\phi(\mathbf{x}) = 4\pi \int d^3x'\, G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')$, or

$$\boxed{\phi(\mathbf{x}) = \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.} \tag{2.12}$$

Using that $\nabla|\mathbf{x} - \mathbf{x}'|^{-1} = -(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^3$, the electric field created by the system of charges is given by

$$\mathbf{E}(\mathbf{x}) = \int d^3x'\, \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{2.13}$$

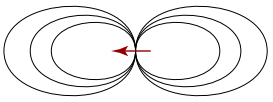
▷ Example: The potential created by a set of N point charges q_i at positions $\mathbf{x}_i$ obtains by integration over $\rho(\mathbf{x}) = \sum_{i=1}^N q_i \delta(\mathbf{x} - \mathbf{x}_i)$,

$$\phi(\mathbf{x}) = \sum_{i=1}^N \frac{q_i}{|\mathbf{x} - \mathbf{x}_i|}. \tag{2.14}$$

▷ Example: Consider an electrostatic **dipole**, i.a. a system of two opposite charges of equal strength $\pm q/a$ at positions $\pm a\mathbf{n}'$, where $\mathbf{n}'$ is a unit vector. (For convenience we place the center of the dipole at the coordinate $\mathbf{x} = 0$.) The corresponding charge distribution is given by $\rho(\mathbf{x}) = \frac{q}{a}(\delta(\mathbf{x}-a\mathbf{n}'/2)-\delta(\mathbf{x}+a\mathbf{n}'/2))$. Assuming that $|\mathbf{x}| \gg a$, the potential created by the dipole is given by

$$\phi(\mathbf{x}) = \frac{q}{a}\left(\frac{1}{|\mathbf{x}-a\mathbf{n}'/2|} - \frac{1}{|\mathbf{x}+a\mathbf{n}'/2|}\right) = \frac{q\mathbf{n}'\cdot\mathbf{x}}{|\mathbf{x}|^3} + \mathcal{O}((a^2/|\mathbf{x}|)^2)$$

In the limit $a \to 0$ — two infinitely charged and infinitesimally separated point charges — the source configuration becomes a point dipole. The product $q\mathbf{n}' \equiv \mathbf{d}$ is denoted its **dipole moment**. The potential created by a point dipole at the coordinate origin is given by



$$\phi(\mathbf{x}) = \frac{\mathbf{d}\cdot\mathbf{x}}{|\mathbf{x}|^3}, \tag{2.15}$$

i.e. it is anisotropic and decays doubly as fast as the potential of a point charge (see the figure for a schematic of the equi–potential lines of a dipole $\mathbf{d}$.) Taking the gradient of this expression, we obtain the electric field of the dipole,

$$\mathbf{E}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{d}) - \mathbf{d}}{|\mathbf{x}|^3}, \tag{2.16}$$

where $\mathbf{n}$ is the unit vector in $\mathbf{x}$–direction.

As a corollary, we note that our analysis above implies the identity,

$$\frac{1}{|\mathbf{x}|} \xrightarrow{\text{F.T.}} \frac{1}{2\pi^2}\frac{1}{k^2}. \tag{2.17}$$

Using this formula we may solve the Poisson equation directly (i.e. without using Green functions) by Fourier transformation techniques: Fourier transformation of $\Delta\phi = -4\pi\rho$ obtains $k^2\phi(\mathbf{k}) = 4\pi\rho(\mathbf{k})$ or $\phi(\mathbf{k}) = 4\pi k^{-2}\rho(\mathbf{k}) = (2\pi)^3(2\pi^2k^2)^{-1}\rho(\mathbf{k})$. Now, we know that $(2\pi)^d$ times the product of two functions in Fourier space transforms to convolution of the real space representations. Thus,

$$\phi(\mathbf{x}) = \left(\frac{1}{|\mathbf{x}|} * \rho\right)(\mathbf{x}) = \int d^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.$$

Energy considerations

There is no a priori rule how to attribute an 'energy' to an electromagnetic field. However, we *do* know of a mechanics-inspired prescription how to compute the energy of a system of (point) charges: Consider a system of N point charges at positions $\mathbf{x}_i$. To a configuration of $N = 1$ charges we assign the energy $E_1 = 0$.

▷ Info. There is, of course, some arbitrariness in assigning energy zero to a single non–vanishing charge in vacuum. This charge will create a finite electric field and it is, somehow, counter–intuitive to treat this system as one void of 'energy'. However, every attempt to meaningfully assign a 'self–energy' to point charges in classical electrodynamics leads to unphysical divergences. The problem of the **infinite**

self–energy finds it resolution only in quantum electrodynamics. All we can do in the classical context is to artificially normalize the infinite constant corresponding to the self–energy to zero.

The mechanic energy (i.e. the work to be done against the force field $\mathbf{F} = q_2\mathbf{E} = -q_2\nabla\phi$, where ϕ is the potential created by charge no.1) required to bring a second charge from infinity to its destination coordinate $\mathbf{x}_2$ is given by $E_2 = q_2\phi(\mathbf{x}_2-\mathbf{x}_1) = q_2q_1/|\mathbf{x}_1-\mathbf{x}_2|$. Bringing in a third charge against the potential of the two charges that have already been placed will cost the additional energy $q_3q_1/|\mathbf{x}_1-\mathbf{x}_3|+q_3q_2/|\mathbf{x}_2-\mathbf{x}_3|$, i.e. the total energy required to build up a three charge configuration is given by $E_3 = q_2q_1/|\mathbf{x}_1-\mathbf{x}_2|+q_3q_1/|\mathbf{x}_1-\mathbf{x}_3|+q_3q_2/|\mathbf{x}_2-\mathbf{x}_3| = \sum_{1\le i<j\le 3} q_iq_j/|\mathbf{x}_i-\mathbf{x}_j|$. Iterating this procedure, we conclude that the mechanic energy of an N–charge configuration is given by

$$E = \sum_{1\le i<j\le N} \frac{q_iq_j}{|\mathbf{x}_i-\mathbf{x}_j|} = \frac{1}{2}\sum_{i\neq j}^{N} \frac{q_iq_j}{|\mathbf{x}_i-\mathbf{x}_j|}. \tag{2.18}$$

Thinking of a continuum charge distribution ρ as a (dense) accumulation of point charges, it is evident that the continuum generalization of this result, i.e. the **energy of a charge distribution** is given by

$$\boxed{E = \frac{1}{2}\int d^3x d^3x' \,\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.} \tag{2.19}$$

▷ Info. However, what happened to the exclusion rule $i \neq j$? The problem of self energies relates to the divergence of the denominators $1/|\mathbf{x}-\mathbf{x}'|$ as the points $\mathbf{x}$ and $\mathbf{x}'$ approach each other. (Gedanken experiment: Think of a point charge q as two charges $q/2$ infinitely closely spaced. The energy of this configuration, computed according to (2.18), diverges.) However, for any continuous charge distribution, the integral (2.19) is finite, i.e. there is no problem with the potentially dangerous regions $\mathbf{x} \simeq \mathbf{x}'$ where the arguments $\mathbf{x}$ and $\mathbf{x}'$ approach each other. To see this, consider the contribution to the integral from regions where $|\mathbf{x}-\mathbf{x}'| < a$ is small:

$$\int\!\!\int_{|\mathbf{x}-\mathbf{x}'|<a} d^3x d^3x' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \simeq \int d^3x\rho^2(\mathbf{x}) \int_{|\mathbf{y}|<a} d^3y\,\frac{1}{|y|} = \int d^3x\rho^2(\mathbf{x})4\pi\int_0^a y^2\frac{dy}{y} = \int d^3x\rho^2(\mathbf{x})2\pi a^2,$$

which is finite: The 'smallness' of the three–dimensional volume element $y^2dy\sin\theta d\theta d\phi$ over–compensates the divergence of the denominator. We thus conclude that (2.19) adequately describes the energy of the continuum charge distribution and that the problem of the self energy does not arise.

An interesting reformulation of the energy formula (2.19). Using Eq. (2.12),

$$E = \frac{1}{2}\int d^dx\,\phi(\mathbf{x})\rho(\mathbf{x}) = -\frac{\epsilon_0}{8\pi}\int d^dx\,\phi(\mathbf{x})\Delta\phi(\mathbf{x}) \overset{(1.10)}{=} \frac{\epsilon_0}{8\pi}\int d^dx\,\nabla\phi(\mathbf{x})\nabla\phi(\mathbf{x}) =$$
$$= \frac{1}{8\pi}\int d^3x\,\mathbf{D}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}),$$

where, for later reference, we have re–installed the dielectricity constant $\epsilon_0 = 1$. Remarkably, the last representation no longer makes reference to the matter–origin of the electric field; it assigns an

energy to the electric field *as such*. We will, thus, use

$$\boxed{E = \frac{1}{8\pi}\int d^3x\, \mathbf{D}\cdot\mathbf{E}.} \tag{2.20}$$

as a definition of the total **energy of the electric field**. The energy density, i.e., the energy per volume element is given by $u_E(\mathbf{x}) \equiv \frac{1}{8\pi}(\mathbf{D}\cdot\mathbf{E})(\mathbf{x})$. Later on we will see that many other characteristics of matter distributions — momentum, angular momentum, etc. — allow for a generalization to the electromagnetic context.

Surface charges and dipole layers

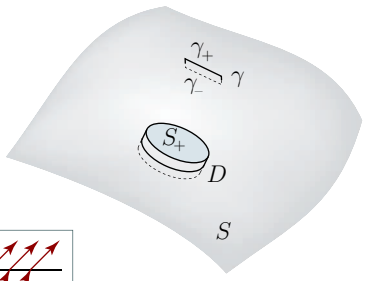
In many applications one is met with charge distributions residing on geometric structures of dimensionality lower than three — charged wires, or surfaces, planar accumulations of dipoles, etc. We here discuss two important subspecies, viz. surface charge distributions and dipole layers.

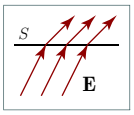
surface charges: Consider a charged quasi–two dimensional surface. We idealize this object by a truly two–dimensional surface, S. (Exercise: invent a limiting process, similar in spirit to our discussion of the point dipole above, whereby the two–dimensional structure emerges from a charge distribution of finite thickness.) We define $\eta(\mathbf{x})d\sigma$ to be the charge residing in the surface element $d\sigma$ at $\mathbf{x}$. The electric field created by the surface can be obtained by obvious generalization of the general formula (2.13):

$$\mathbf{E}(\mathbf{x}) = \int_S d\sigma\, \eta(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{2.21}$$

While Eq.(2.21) fully solves the problem, it does not tell us immediately how the electric field will actually behave in the immediate vicinity of the surface. Information of this type is conveniently obtained by running a few 'tests' based on the integral form of the Maxwell equations:

To this end, consider a little pill box D whose upper lid, S_+ (lower lid, S_-) is above (beneath) the surface S and whose height is assumed to be vanishingly small. We assume that the two surfaces $S_\pm$ are adjusted parallel to S and that they are so small that variations of the electric field across $S_\pm$ are negligible. Under these conditions, the surface integral of $\mathbf{E}$ over D obtains $\int_D \mathbf{E} \simeq \delta S(\mathbf{E}(\mathbf{x}_+) - \mathbf{E}(\mathbf{x}_-))\cdot\mathbf{n} \equiv \delta S \Delta\mathbf{E}_\perp(\mathbf{x})$, where δS is the area of $S_\pm$, $\mathbf{x}_\pm \in S_\pm$ are points infinitesimally above (below) the surface, $\mathbf{n}$ is the unit normal vector to S (and $S_\pm$) at $\mathbf{x}$, and $\Delta\mathbf{E}_\perp(\mathbf{x})$ denotes the jump in the normal component of $\mathbf{E}$ at $\mathbf{x}$. At the same time, by the Maxwell equation (1.23), the surface integral equals $\int_{V(D)} d^3x\, \nabla\cdot\mathbf{E} = 4\pi\int_{V(D)} d^3x\,\rho(\mathbf{x}) = 4\pi\int_{V(D)\cap S} d\sigma\eta(\mathbf{x}) \simeq 4\pi\delta S\eta(\mathbf{x})$. Comparing the two results we find





$$\Delta\mathbf{E}_\perp(\mathbf{x}) = 4\pi\eta(\mathbf{x}), \tag{2.22}$$

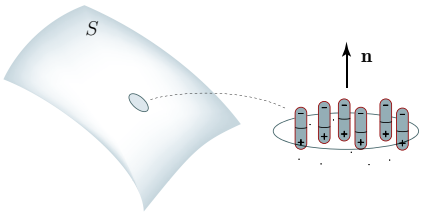
i.e. the component of the electric field normal to the surface jumps by an amount set by the surface charge.

To understand the behaviour of the tangential component, $\mathbf{E}_\parallel$, we integrate $\mathbf{E}$ over a small closed contour γ comprising two stretches $\gamma_\pm$ infinitesimally above/below S. Computing the line integral under the same assumptions as above, we obtain $\oint_\gamma d\mathbf{s}\cdot\mathbf{E} \simeq l(\mathbf{E}_\parallel(\mathbf{x}_+) - \mathbf{E}_\parallel(\mathbf{x}_-))$, where

l denotes the length of $\gamma_\pm$ and the contribution to the line integral from the stretches normal to S has been neglected. However, by Eq. (**??**), (or, equivalently, the condition $\nabla \times \mathbf{E} = 0$), this integral equals zero, i.e.

$$\Delta \mathbf{E}_\parallel(\mathbf{x}) = 0, \tag{2.23}$$

the tangential component of the electric field is continuous across S. Altogether, we obtain a profile of $\mathbf{E}$ as indicated qualitatively in the figure.



dipole layers: Dipole layers are quasi two–dimensional configurations S of stacked dipoles. The individual dipole moments $\mathbf{d} \parallel \mathbf{n}$ are oriented normal to the surface (and uniformly oriented, $\mathbf{n} \cdot \mathbf{x} > 0$, say.)

▷ Info. Dipole layers are systems of considerable applied relevance: many **organic or biological molecules** carry a non–vanishing dipole element. Membranes formed by an effectively two–dimensional planar accumulation of such molecules represent dipolar layers.

Let $\eta(\mathbf{x}')\mathbf{n}d\sigma$ be the dipole moment carried by a surface element $d\sigma$ at $\mathbf{x}'$. According to (2.15), the potential created by this element is given by $d\phi(\mathbf{x}) = d\sigma\, \eta(\mathbf{x}') \frac{\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$. The full potential created by the layer is given by

$$\phi(\mathbf{x}) = \int_S d\sigma\, \eta(\mathbf{x}') \frac{\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}.$$

For a discussion of the potential ϕ, we refer to [1].

2.2.3 Poisson equation in non–cartesian coordinates

Whenever a problem in electrostatics possesses an in–built symmetry — rotational symmetry, cylindrical symmetry, etc. — its solution simplifies dramatically. To benefit maximally from the presence of a symmetry one ought to formulate the Poisson equation in problem adjusted coordinates, i.e. a coordinate system in which one or several coordinates remain invariant under the action of the symmetry operation. Coordinate systems of outstanding practical importance are the spherical and cylindrical coordinates.

We next formulate and solve the Poisson and Laplace equation in spherical coordinates. This discussion will provide the basis of the multipole expansion, a concept of importance in numerous applications in science and engineering ...

▷ Math. To prepare the analysis below, we need to introduce the notion of **complete systems of orthonormal functions**, a concept of great practical relevance in electrodynamics (plus quantum mechanics and many other areas of physics for the matter.)

A brief reminder of linear algebra: consider an n–dimensional real vector space V with scalar product $V \times V \to \mathbb{R}, (\mathbf{v}, \mathbf{w}) \mapsto \langle \mathbf{v}, \mathbf{w} \rangle$. Let $A : V \to V$ be an operator that is symmetric w.r.t. $\langle\ ,\ \rangle$, i.e. $\forall \mathbf{v}, \mathbf{w} \in V : \langle \mathbf{v}, A\mathbf{w} \rangle = \langle A\mathbf{v}, \mathbf{w} \rangle$. Then, (i) A can be diagonalized (by an orthogonal transformation), i.e. there exists a basis of eigenstates $\mathbf{v}_a$, $a = 1, \ldots, n$ with $A\mathbf{v}_a = \lambda_a \mathbf{v}_a$. (ii) We may chose the $\mathbf{v}_a$'s to be mutually orthogonal and normalized: $\langle \mathbf{v}_a, \mathbf{v}_b \rangle = \delta_{ab}$, i.e. $\{\mathbf{v}_a\}$ is an orthonormal basis. (iv) An arbitrary

element $\mathbf{v} \in V$ can be expanded in the basis $\{\mathbf{v}_a\}$: $\mathbf{v} = \sum_{i=a}^{n} c_a \mathbf{v}_a$, where the coefficients c_a are given by $c_a = \langle \mathbf{v}_a, \mathbf{v} \rangle$.

Remarks: (a) The eigenbasis of A can be obtained by the Gram–Schmidt orthogonalization algorithm. (b) Nowhere do we require that the scalar product be the trivial one. I.e. $\langle \mathbf{v}, \mathbf{w} \rangle = v_i g_{ij} w_j$, where g_{ij} may be an arbitrary symmetric non–degenerate matrix. (d) Keep in mind: whenever we say "An operator A is symmetric", what we mean, in fact, is "it is symmetric w.r.t. a specifically chosen scalar product". This scalar product can, but need not, be the standard one. Only in that special case does the 'symmetric' operator assume the form of a symmetric matrix.

In functional analysis it is shown that, and under which conditions, the statements above afford a generalisation to the case of infinite dimensional vector spaces. We will here naively[5] formulate this generalisation for those infinite dimensional spaces that are the most important with regard to applications in physics: function spaces.

As mentioned earlier, the space of differentiable[6] functions $\mathcal{C}(U, \mathbb{R})$ defined on some subset $U \subset \mathbb{R}^d$ forms a vector space. In addition to differentiability we will usually also require integrability, i.e. we will consider functions $f \in L^2(U)$, where $L^2(U)$ is the space of square integrable functions ($\int_U d^d x\, f(\mathbf{x})^2$ exists.) To keep the notation slim we will denote the intersection $L^2(U) \cap \mathcal{C}(U, \mathbb{R})$ again by $L^2(U)$. Remark: to bridge the gap between function spaces and finite dimensional vector spaces, imagine a function as the limit of a finite–dimensional object: We may approximate any continuous function $f : U \to \mathbb{R}$ by introducing a fine discretization (a 'grid') in U and assigning to each of its base points $\mathbf{x}_i \in U$ the discrete value $f(\mathbf{x}_i)$. In the limit of an infinitely dense grid, the discrete vector formed by all the values $\{f(\mathbf{x}_i)\}$ contains information equivalent to the full function. We may, thus, approximate any continuous functions by a sufficiently high–dimensional conventional vector. This analogy is often useful.

Now, the vector space $L^2(U)$ can be equipped with a real scalar product. Indeed, it is straightforward to verify that

$$\begin{aligned} \langle\ ,\ \rangle : L^2(U) \times L^2(U) &\to \mathbb{R}, \\ (f, g) &\mapsto \langle f, g \rangle \equiv \int_U d^d x\, f(\mathbf{x}) g(\mathbf{x}) \end{aligned}$$

satisfies the defining equations of a **scalar product**. (Do it!)

Importantly, however, the 'standard' scalar product above is not always the natural one. By way of example, consider the space of functions defined on the two–sphere S^2. Let us parameterize S^2 by the standard set of coordinates, $U = \{(\theta, \phi) \in [0, \pi[\times [0, 2\pi[\}$.[7] The 'natural' scalar product on the two–sphere is given by

$$\langle f, g \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, f(\theta, \phi) g(\theta, \phi), \tag{2.24}$$

i.e. it accounts for the surface area element of the sphere.[8]

Now, consider a linear differential operator D acting in $L^2(U)$. (Alluding to the discretised picture above, we may think of D as a high–dimensional matrix; approximating derivative operations by finite

[5] ... means without discussing the crucial convergence criteria required to firmly establish the diagonalisability of symmetric operators.

[6] In a somewhat sloppy notation we will use the symbol $\mathcal{C}$ to denote spaces of functions that are as often differentiable as is required by the problem at hand.

[7] Critical readers will object that U does not parameterize the full sphere; a line from the north pole to the south pole is excluded. This, however, does not play a role in our present discussion.

[8] Yet more generally, one might even consider scalar products of the form $\langle f, g \rangle = \int d^d x d^d x'\, f(\mathbf{x}) \tilde{g}(\mathbf{x}, \mathbf{x}') g(\mathbf{x}')$, where $\tilde{g}(\mathbf{x}, \mathbf{x}') = \tilde{g}(\mathbf{x}', \mathbf{x})$ is a symmetric function of two arguments. However, constructs of this type usually do not appear in applications.

differences, D does, literally, assume the form of a matrix acting on the discrete approximation of functions.) An interesting situation arises if D is symmetric w.r.t. the scalar product on the function space, i.e. if $\langle f, Dg\rangle = \langle Df, g\rangle$, or

$$\int d^d x\, f(\mathbf{x})\tilde{g}(\mathbf{x})(Dg)(\mathbf{x}) = \int d^d x\, (Df)(\mathbf{x})\tilde{g}(\mathbf{x})g(\mathbf{x}),$$

where $\tilde{g}$ is the function defining the metric. Under these circumstances, the operator D possesses a complete system of **eigenfunctions** $\{f_n\}$, i.e. functions obeying the (differential) equation $Df_n(\mathbf{x}) = \lambda_n f_n(\mathbf{x})$. These functions are mutually orthogonal and can be normalized: $\langle f_n, f_{n'}\rangle = \delta_{nn'}$. Functions $f \in L^2(U)$ can be expanded in the set $\{f_n\}$,[9]

$$f(\mathbf{x}) = \sum_n \langle f, f_n\rangle\, f_n(\mathbf{x}).$$

▷ Example: Let $U = [0, 1]$. We consider the space of functions $L^2(U)$ subject to the constraint $f(0) = f(1) = 0$ ('Dirichlet boundary conditions'). Let $\langle f, g\rangle \equiv \int_0^1 dx\, f(x)g(x)$ be the standard metric and consider the differential operator $D = -d_x^2$. The eigenfunctions of this operator are given by $f_n(x) = \sqrt{2}\sin(\pi n x)$. They are orthonormalized, $\int_0^1 dx\, f_n(x)f_{n'}(x) = \delta_{nn'}$ and the expansion

$$f(x) = \sqrt{2}\sum_n c_n \sin(\pi n x), \qquad c_n = \sqrt{2}\int_0^1 dx\, f(x)\sin(\pi n x)$$

is the familiar **Fourier series** expansion.

▷ Example: Let $U = [-1, 1]$ and $D = -d_x(1 - x^2)d_x$. The operator D is symmetric w.r.t. the standard scalar product on U. (Check it!) Its eigenfunctions $DP_l(x) = \lambda_l P_l(x)$ are known as **Legendre polynomials**. The first few Legendre polynomials are given by

$$\begin{aligned} P_0(x) &= 1 & P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) & P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) & &\ldots. \end{aligned}$$

The generalization to polynomials of arbitrary degree is given by **Rodrigues' formula**

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l, \qquad \lambda_l = l(l + 1). \tag{2.25}$$

Breaking with the general rule of scalar product normalization, the Legendre polynomials are normalized by $P_l(1) = 1$.

All we said below generalizes to the case of complex vector spaces. A **complex scalar product** on a finite dimensional complex vector space is defined by $\langle \mathbf{v}, \mathbf{w}\rangle = v_i^* g_{ij} w_j$, where $g = \{g_{ij}\}$ is an hermitian matrix: $g = g^\dagger$. Notice that $\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle^*$. A general hermitian operator A, $\langle A\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, A\mathbf{w}\rangle$

[9] In cases where U is an unbounded set, the set of eigenvalues $\{\lambda_n\}$, the so–called **spectrum** of the operator D, may become (partially) dense. In this case, denoting the eigenfunctions $f(k)$, where k is a continuous index, the expansion assumes the form of an integral $f(\mathbf{x}) = \int dk \langle f(k), f\rangle f(\mathbf{x})$. The orthonormalisation condition assumes the form $\langle f(k), f(k')\rangle = \delta(k - k')$. (Why?)

can (i) be diagonalized by an unitary transformation, i.e. there exists a basis of eigenstates $\{\mathbf{v}_a\}$, with $A\mathbf{v}_a = \lambda_a \mathbf{v}_a$, where $\lambda_a \in \mathbb{R}$. Again, we may chose these states to be orthonormal $\langle \mathbf{v}_a, \mathbf{v}_b \rangle = \delta_{ab}$. An arbitrary element $\mathbf{v} \in V$ can be expanded in the basis $\{\mathbf{v}_a\}$: $\mathbf{v} = \sum_{i=a}^{n} c_a \mathbf{v}_a$, where the coefficients c_a are given by $c_a = \langle \mathbf{v}_a, \mathbf{v} \rangle$.

The generalization to the infinite dimensional case is obvious: consider the **complex function space**, $L^2(U, \mathbb{C})$, i.e. the space of square integrable functions $f : U \to \mathbb{C}$. We define a scalar product by

$$\langle f, g \rangle = \int_U d^d x \, \tilde{g}(\mathbf{x}) f^*(\mathbf{x}) g(\mathbf{x}),$$

where the weight function $\tilde{g}$ is real (why? Think of $\tilde{g}$ as the generalization of a diagonal hermitian matrix.) A linear differential operator D is hermitian if $\forall f, g \in L^2(U, \mathbb{C}) : \langle f, Dg \rangle = \langle Df, g \rangle$. The eigenfunctions, f_n, of an hermitian differential operator form an (orthonormalizable) complete set. An arbitrary function can be expanded in the basis $\{f_n\}$ as $f = \sum \langle f_n, f \rangle \, f_n$.

▷ Example: On the interval $U = [0, 2\pi[$ consider the space of functions $f : U \to \mathbb{C}$ with 'periodic boundary conditions', $f(0) = f(x \to 2\pi)$. We chose the standard complex scalar product $\langle f, g \rangle = \int_0^{2\pi} dx \, f^*(x) g(x)$. The orthonormalized eigenfunctions of the differential operator $D = -d_x^2$, $f_n(x) = (2\pi)^{-1/2} \exp(inx)$, $n \in \mathbb{Z}$ form a complete set. The expansion

$$f(x) = \frac{1}{(2\pi)^{1/2}} \sum_n c_n e^{inx}, \qquad c_n = \frac{1}{(2\pi)^{1/2}} \int dx \, e^{-inx} f(x)$$

is the familiar **Fourier series expansion** on U.

The table below summarizes a number of relevant identities on linear symmetric operators in finite and infinite–dimensional vector spaces.

	finite dimensional	function spaces
real scalar product	$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_i v_i g_{ij} w_j$	$\langle f, g \rangle = \int d^d x \, \tilde{g}(\mathbf{x}) f(\mathbf{x}) g(\mathbf{x})$
complex scalar product	$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_i v_i^* g_{ij} w_j$	$\langle f, g \rangle = \int d^d x \, \tilde{g}(\mathbf{x}) f^*(\mathbf{x}) g(\mathbf{x})$
sym. of linear operator	$\langle \mathbf{v}, A\mathbf{w} \rangle = \langle A\mathbf{v}, \mathbf{w} \rangle$	$\langle f, Dg \rangle = \langle Df, g \rangle$
eigenvector/eigenfunction	$A\mathbf{v}_a = \epsilon_a \mathbf{v}_a$	$Df_n = \lambda_n f_n$
orthonormalisation	$\langle \mathbf{v}_a, \mathbf{v}_b \rangle = \delta_{ab}$	$\langle f_n, f_m \rangle = \delta_{nm}$
expansion	$\mathbf{v} = \sum_a \langle \mathbf{v}_a, \mathbf{v} \rangle \, \mathbf{v}_a$	$f = \sum \langle f_n, f \rangle \, f_n$

Table 2.2: Identities on complete set of states in finite and infinite dimensional vector spaces.

Solution of the Laplace equation (spherical coordinates)

Consider the Laplace operator in spherical coordinates,

$$\Delta = \Delta^{\#} + r^{-2} \Delta^s,$$

$$\Delta^{\#} = \frac{1}{r}\partial_r^2\, r, \tag{2.26}$$

$$\Delta^{s} = \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2, \tag{2.27}$$

where the derivatives act on everything to the right. Our first objective is to compute the general solution of the Laplace equation

$$\Delta\Phi(r,\theta,\phi) = 0,$$

for unspecified boundary conditions. In spite of its formidable appearance, the Laplace operator has one weak spot: it is **separable**, i.e. the sum of three pieces each of which contains only derivatives w.r.t. one of the three coordinates. This suggests to represent the solution as a product,

$$\Phi(r,\theta,\phi) = U(r)P(\theta)Q(\phi).$$

Substituting this ansatz into the Laplace equation and multiplying by r^2/UPQ, we obtain

$$U^{-1}(r^2\Delta^{\#})U = -(PQ)^{-1}\Delta^{s}PQ.$$

Now the l.h.s./r.h.s. of this equation depends only on the variables $r/(\theta,\phi)$. The only way to satisfy this condition (think about it!) is to require that the two sides of the equation are equal *and* constant. Denoting this constant by $l(l+1)$,[10] we obtain the two equations

$$\Delta^{\#}U = l(l+1)r^{-2}U, \tag{2.28}$$

$$\Delta^{s}PQ = -l(l+1)PQ. \tag{2.29}$$

The first of these equations is controlled by the **radial component of the Laplace operator**. Substituting the monomial ansatz $U \sim r^{\alpha}$ into the radial equation, we obtain the solution

$$U = a^{+}r^{l} + a^{-}r^{-(l+1)}. \tag{2.30}$$

The presence of two undetermined constants signals that we have, indeed, found the most general equation of the (second order) differential equation for U. Notice that the first/second term on the r.h.s. diverges for $r \to \infty/r \to 0$, i.e. the choice of the constants $a^{\pm}$ will be dictated by the boundary conditions imposed on the Laplace equation for large/small values of the radial coordinate.

Turning to the **angular component of the Laplace operator**, we may apply an argument similar to the one used above to separate the problem into the solution of two ordinary differential equations: multiplication of Eq. (2.29) by $\sin^2\theta PQ$ obtains

$$P^{-1}(\sin\theta\partial_\theta\sin\theta\partial_\theta + l(l+1)\sin^2\theta)P = -Q^{-1}\partial_\phi^2 Q,$$

i.e. an equation whose l.h.s./r.h.s. depends only on θ/ϕ. Requiring constancy of both sides, we obtain the two ordinary differential equations

$$d_\phi^2 Q + m^2 Q = 0, \tag{2.31}$$

$$\left[(\sin\theta)^{-1}d_\theta\sin\theta d_\theta + l(l+1) - m^2(\sin\theta)^{-2}\right]P = 0, \tag{2.32}$$

[10] Later on, we shall see that l is integer. The positivity of the constant $l(l+1) \geq 0$ is required by the fact that the operator $-r^2\Delta^{\#}$ (regarded as an operator acting in the space of square integrable functions) is positive definite.

where $m^2 = \text{const.}$. The first of these equations is solved by $Q = \exp(im\phi)$. In order for Q to be single–valued, $Q(\phi \to 2\pi) = Q(0)$, we must require $m \in \mathbb{Z}$.[11] Changing variables, $x = \cos\theta$, the remaining, polar equation assumes the form

$$d_x \left[(1-x^2)d_x P\right] + l(l+1) - (1-x^2)^{-1}m^2\big] P = 0.$$

For $m = 0$, this equation collapses to the differential equation $[d_x(1-x^2)d_x + l(l+1)]P = 0$ discussed in an example on p xxxiv; its solutions are the Legendre polynomials $P_l(x) \equiv P_l^{m=0}$.

As for the general case, $m \neq 0$, it turns out that solutions that are finite on the interval $[-1, 1]$ exist only if (a) l is integer and (b) limited to values $l \geq |m|$. The solutions corresponding to the allowed values $m = -l, -l+1, \ldots, l$ of the azimuthal index, P_l^m, are called **Legendre functions**. For positive m they are defined by

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x). \tag{2.33}$$

The solutions for negative values of m are given by $P_l^{m<0} = (-)^m \frac{(l-m)!}{(l+m)!} P_l^m$.

Summarizing, we have found that the so–called **spherical harmonics**

$$\boxed{Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\exp(im\phi)} \tag{2.34}$$

solve the angular part of the problem, $\Delta^s Y_{lm} = -l(l+1)Y_{lm}$. A number of remarks on this important family of functions:

We first note that the angular part of the Laplace operator, Δ^s is symmetric w.r.t. the natural scalar product on the two–sphere, (2.24). (prove it!) This means that its eigenfunctions, viz. the family of functions $Y_{lm} \propto P_l^m(\cos\theta)Q_m(\phi)$ form a complete set of orthogonal functions on the sphere. Indeed, one can prove that

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, Y_{lm}^*(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{2.35}$$

The unit–normalization of the r.h.s. determines the normalization factor in (2.34). The eigenvalues $\lambda_l = -l(l+1)$ are $(2l+1)$–fold degenerate, i.e. each Y_l^m, $m = -l, \ldots, l$ has eigenvalue $l(l+1)$. Arbitrary functions on the sphere can be expanded in spherical harmonics as[12]

$$f(\theta,\phi) = \sum_{lm} a_{lm}\, Y_{lm}(\theta,\phi), \qquad a_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, Y_{lm}^*(\theta,\phi) f(\theta,\phi). \tag{2.36}$$

Eq. (2.36) entails the completeness relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) = \delta(\phi-\phi')(\sin\theta)^{-1}\delta(\theta-\theta'). \tag{2.37}$$

[11] Notice that the set of functions $\{Q_m\}$ is a complete set on the interval $[0, 2\pi]$, cf. the example on p xxxv.

[12] The possibility to expand functions in this way motivates the denotation 'spherical harmonics': 'spherical' for we are on the sphere, 'harmonics' because the functions Y_{lm} assume the role of the harmonic functions $\sim \exp(ikx)$ on flat integration spaces.

For the sake of concreteness, the explicit form of a few spherical harmonics is given below:

$$\begin{aligned}
l=0 \quad &: \quad Y_{00} = \frac{1}{\sqrt{4\pi}}, \\
l=1 \quad &: \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \\
&\qquad Y_{1,\pm 1} = \sqrt{\frac{3}{8\pi}}\sin\theta\ \exp(\pm i\phi), \\
l=2 \quad &: \quad Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \\
&\qquad Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\ \exp(\pm i\phi), \\
&\qquad Y_{2,\pm 2} = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta\ \exp(\pm 2i\phi).
\end{aligned} \tag{2.38}$$

For later reference, we notice that

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta), \qquad Y_{lm}(0,\phi) = \delta_{m,0}\sqrt{\frac{2l+1}{4\pi}}. \tag{2.39}$$

Combining Eq.(2.34) with our earlier result (2.30) for the radial function, we conclude that the **general solution of the Laplace equation in spherical coordinates** is given by

$$\boxed{\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a^{+}_{lm}r^{l} + a^{-}_{lm}r^{-(l+1)}\right)Y_{lm}(\theta,\phi),} \tag{2.40}$$

where the value of the constants $a^{\pm}$ is determined by the boundary conditions of the problem.

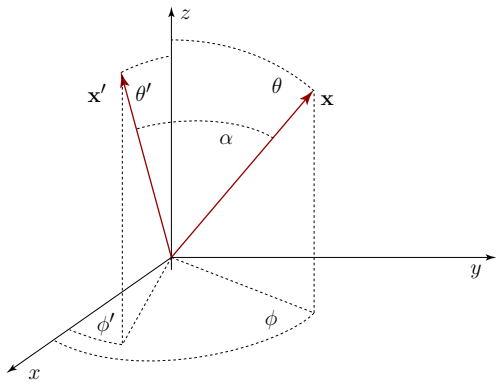
Addition theorem

At this point, it is a good idea to take a step back and ask where our so far analysis has got us: Firstly, we have not yet solved the Poisson equation — the prime information carrier on a source–ful system — in spherical coordinates. Second, it has not yet become clear enough why it is worthwhile to invest the considerable effort to formulate the theory in spherical coordinates at all. In this and the following section, we will address these two remaining issues in turn.

Rather than solving the **Poisson equation** directly in spherical coordinates, we will build on previous results, i.e. use that the Green function of electrostatics (in unbounded space) is given by Eq.(2.9). In the math block below we prove that the Green function of infinite–space electrostatics, characteristic distance function $G(\mathbf{x},\mathbf{x}') = (4\pi|\mathbf{x}-\mathbf{x}'|)^{-1}$ affords the expansion

$$\boxed{G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\frac{r_{<}^{l}\,r_{>}^{-(l+1)}}{2l+1}\sum_{m=-l}^{l}Y^{*}_{lm}(\theta,\phi)Y_{lm}(\theta',\phi'),} \tag{2.41}$$

where $r_</r_>$ denotes the smaller/larger of the radial arguments r and r'. Eq.(2.41) is known as the **addition theorem**. In the next section we will apply it to demonstrate the utility of the spherical harmonics in applied electrostatics.



▷ Math. Before turning to the actual **proof of the addition theorem**, notice that the structure of the r.h.s. of Eq. (2.41) is essentially fixed by the condition that for $\mathbf{x} \neq \mathbf{x}'$ the l.h.s. solves the Laplace equation, $\Delta_x|\mathbf{x}-\mathbf{x}'|^{-1} = \Delta_{x'}|\mathbf{x}-\mathbf{x}'|^{-1} = 0$, where $\Delta_{x/x'}$ is the Laplace operator acting on the argument $\mathbf{x}/\mathbf{x}'$. This means that as a function of both arguments $\mathbf{x}$ and $\mathbf{x}'$, the r.h.s. must be of the structure (2.40). To avoid divergences at large and small values of the arguments, resp., the expansion in the larger/smaller of the two radial coordinates must not engage the dangerous powers $r^l/r^{-(l+1)}$. This fixes the structure of the radial term $r_<^l r_>^{-(l+1)}$.

However, what the schematic argument above cannot tell us is the precise form of the expansion coefficients. To actually prove the theorem, we consider the Green function of electrostatics as defined by (2.8). Let us assume that G admits an expansion of the type

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l(r,r')Y_{lm}^*(\theta,\phi)Y_{lm}(\theta',\phi'). \tag{2.42}$$

Substituting this ansatz into the defining equation, using the polar representation, $\Delta = \Delta^{\#} + r^{-2}\Delta^s$, and that $\Delta^s Y_{lm} = -l(l+1)Y_{lm}$ we obtain

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big(\Delta^{\#} - r^{-2}l(l+1)\Big)\, g_l(r,r')Y_{lm}^*(\theta,\phi)Y_{lm}(\theta',\phi') = -\delta(\mathbf{x}-\mathbf{x}').$$

Now, $\delta(\mathbf{x}-\mathbf{x}') = (r-r')^{-2}\delta(r-r')\delta(\phi-\phi')(\sin(\theta-\theta'))^{-1}\delta(\theta-\theta') \overset{(2.37)}{=} r^{-2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^*(\theta',\phi')Y$
Substituting this representation of the δ–function, we obtain

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big[\Big(\Delta^{\#} - r^{-2}l(l+1)\Big)\, g_l(r,r') + \delta(r-r')\Big]\, Y_{lm}^*(\theta,\phi)Y_{lm}(\theta',\phi') = 0.$$

The completeness of the functions Y_{lm} implies[13] the vanishing of the 'coefficients' $[\ldots]$ for all values of (l,m) separately:

$$\Big(-\Delta^{\#} + r^{-2}l(l+1)\Big)\, g_l(r,r') = \delta(r-r').$$

Consider this equation for $r \neq r'$. Then (cf. Eq.(2.30)), the function $g_l(r,\,.\,)$ must be a solution of the radial Laplace equation. We thus conclude that

$$\begin{aligned} r < r' : \quad & g_l(r,r') = a_{1,l}^+ r^l + a_{1,l}^- r^{-(l+1)}, \\ r > r' : \quad & g_l(r,r') = a_{2,l}^+ r^l + a_{2,l}^- r^{-(l+1)}, \end{aligned}$$

[13] This can be seen by multiplying the equation by $Y_{l'm'}^*(\theta'\phi')$ and integrating (θ',ϕ') over the spherical measure. Eq.(2.35) then implies $[\ldots]Y_{lm}(\theta,\phi) = 0$ which means that the prefactor $[\ldots] = 0$.

where the coefficients will be functions of the second argument, r'. (Due to the presence of a δ–function on the r.h.s. of the differential equation, we do not expect the coefficients to be continuous across the point $r = r'$.) Since, however, the coefficients do not depend on the angular variables, we may determine their value by considering specific configurations of $(\theta, \ldots, \phi')$. Let us choose, then, $\theta = \ldots = \phi' = 0$. (Vectors $\mathbf{x}$ and $\mathbf{x}'$ aligned and pointing in z–direction.) Then $|\mathbf{x} - \mathbf{x}'|^{-1} = |r - r'|^{-1}$ becomes a scalar distance function. For both $r > r'$ and $r < r'$, this function be expanded in a power series: $|r - r'|^{-1} \stackrel{r>r'}{=} r^{-1} \sum_{l=0}^{\infty} (r'/r)^l$ and $|r - r'|^{-1} \stackrel{r<r'}{=} r'^{-1} \sum_{l=0}^{\infty} (r/r')^l$. Substituting these series into the ansatz (2.42), using (2.39), and (2.9), we obtain

$$\frac{1}{4\pi} \sum_l r^l r'^{-(l+1)} = \sum_l (a^+_{1,l} r^l + a^-_{1,l} r^{-(l+1)}) \frac{2l+1}{4\pi},$$

and a similar formula for $r > r'$. Comparison of the two sides of the equation leads to the identification $a^+_{1,l} = r'^{-(l+1)}/(2l+1)$, $a^-_{2,l} = r'^l/(2l+1)$ and $a^+_2 = a^-_1 = 0$. Summarizing these results in a single formula, we arrive at

$$g_l(r, r') = \frac{r^l_< r^{-(l+1)}_>}{2l+1}.$$

Finally, substituting the function g back into our ansatz for the Green function and remembering Eq.(2.9) we obtain the addition theorem.

2.2.4 Multipole Expansion

Figure 2.1: A charge/observation point configuration that qualifies for a multipole expansion. Discussion, see text.

Imagine we watch a spatially localized charge distribution ρ from a very remote point $\mathbf{x}$. ('Remote' means that $|\mathbf{x}| \gg \max\{|\mathbf{x}'||\rho(\mathbf{x}') \neq 0\}$, where it is assumed that the origin of the coordinate system is chosen somewhere in the support of ρ, i.e. we assume that $|\mathbf{x}|$ by far exceeds the extent of the charge distribution, cf. Fig. 2.1.) The first thing we will notice is, of course, the total charge carried by the distribution, $Q = \int d^3x' \rho(\mathbf{x}')$. Approaching the distribution or, equivalently, increasing the resolution at which we monitor its potential, we might notice that the charge is spread out inhomogeneously in space. E.g. even an overall neutral object might be intrinsically polarized. In a sense to be made precise below, we might, thus, attribute some vectorial dipole moment to the distribution. Approaching it further, we will be able to resolve yet finer details in the spatial pattern of charge. This simple picture suggests that the potential created by a charge distribution might be organized in a series whose individual terms contain information on the spatial structure of the charge at (a) an increasing level of detail whilst (b) decaying the more rapidly in space the higher their information content is. A series of this type indeed exists and it is called the **multipole expansion**.

The derivation of the multipole expansion is based on the addition theorem discussed in the previous section: Substituting Eq.(2.41) into (2.12), we obtain

$$\phi(\mathbf{x}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r^{-(l+1)}}{2l+1} Y_{lm}(\theta,\phi)\, q_{lm}, \tag{2.43}$$

$$q_{lm} = \int_0^\infty r'^2 dr' \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta'\, \rho(r',\phi',\theta')\, r'^l Y^*_{lm}(\theta',\phi'),$$

where the coefficients q_{lm} are the **multipole moments** of the charge distribution. Eq. (2.43) is the series we have been looking for: The potential has been represented as a series of ever more rapidly (the factor $r^{-(l+1)}$) decaying contributions. The entire information on the charge distribution is encoded in the multipole coefficients q_{lm}. The number of these coefficients, and hence the information carried on the structure of the charge distribution, grows as $2l+1$ with l.

To better understand the physical meaning of the series, let us investigate the first few contributions in more detail:

Since $Y_{00} = (4\pi)^{-1/2}$, the first contribution to the multipole expansion is given by

$$\phi^{(0)}(\mathbf{x}) = (4\pi)^{1/2} \frac{q_{00}}{r},$$

where the **monopole moment** $q_{00} = (4\pi)^{-1/2} Q$ is but ($(4\pi)^{-1/2}\times$) the integrated charge of the distribution.

To understand the meaning of the $l = 1$ term, let us define the **dipole moment** of the charge distribution as

$$\mathbf{d} \equiv \int d^3x'\, \rho(\mathbf{x}')\mathbf{x}'. \tag{2.44}$$

Evidently, the vector $\mathbf{d}$ generalizes the dipole moment introduced in the example on p xxviii to arbitrary distributions. Now, from Eq. (2.38) it follows that

$$\begin{pmatrix} q_{1,-1} \\ q_{1,0} \\ q_{1,+1} \end{pmatrix} = \sqrt{\frac{3}{4\pi}} U \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix}$$

i.e. up to the unitary transformation U, the three component vector $\mathbf{q}_1 \equiv \{q_{1,m}\}$ is equivalent to the dipole moment of the distribution. Using that $rY_{1,m} = (3/4\pi)^{1/2} U^*_{mm'} x_{m'}$, and the unitarity of the matrix U, $UU^\dagger = \mathbf{1}$, we then find

$$\phi^{(1)}(\mathbf{x}) = \frac{4\pi}{3} \sum_{m=-1}^{1} r^{-2} Y_{1,m}(\theta,\phi) q_{1,m} = r^{-3} (U^*_{mn} x_n)(U_{mn'} d_{n'}) = \frac{\mathbf{d}\cdot\mathbf{x}}{r^3},$$

i.e. the generalization of the dipole formula (2.15) to a general distribution.

Now, it is straightforward to construct charge distributions that carry neither monopole nor dipole moments (see the figure for an example.) For such distributions, the dominant contribution to the potential is provided by the **quadrupole moments**. $q_{2,m} = \int d^3x' \rho(\mathbf{x}') r'^2 Y_{2,m}$. Using (2.38) to express the spherical harmonics in cartesian coordinates one finds, e.g.,

$$q_{22} = \sqrt{\frac{15}{32\pi}} \int d^3x'\, (x'_1 - i x'_2)^2 \rho(\mathbf{x}'),$$

i.e. the integral of a weight function quadratic in the components of $\mathbf{x}'$ multiplied by the charge distribution. The four other quadrupole moments can be found in a similar manner. (Due to $Y_{lm} = Y^*_{l-m}$ it suffices to compute the coefficients q_{lm} for positive m.) The next order of the multipole moments, the **octupole moments** $q_{3,m}$ create potentials decaying as r^{-4} and are, therefore, of lesser applied relevance. (The same applies to yet higher orders in the expansion.)

To summarize the utility of the multipole expansion,

- ▷ it allows one to express the potential $\phi(\mathbf{x})$ created by a charge distribution in relatively simple terms, viz. as a function of $\mathbf{x}$ rather than by the general integral (2.12) (which has to be calculated for each point $\mathbf{x}$ anew.)
- ▷ The multipole moments can be used to conveniently characterize the characteristics of a charge distribution. E.g. for a nucleus of approximately ellipsoidal shape, the dipole and quadrupole moments carry information on its deviations from a perfectly spherical geometry, etc.
- ▷ Notice, however, that the value obtained for the multipole moments depend on the choice of origin of the coordinate system. For example, for a point charge at $\mathbf{x}_0$, all multipole moments will be non–vanishing unless we chose $\mathbf{x}_0$ to be the origin of the coordinate system (which would, of course, be the most reasonable choice.)

▷ Info. In practice, the most widely used form of the multipole expansion is based on the spherical harmonics. In principle, however, the organization of the potential in a series of ever more rapidly decaying contributions can alternatively be achieved by straightforward Taylor expansion of (2.12). Again assuming that $|\mathbf{x}| \gg |\mathbf{x}'|$ for all points $\mathbf{x}'$ with $\rho(\mathbf{x}') \neq 0$, we may expand the distance function as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{x_i}{r^3}x'_i + \frac{1}{2}\frac{3x_ix_j - r^2\delta_{ij}}{r^5}x'_ix'_j + \dots . \tag{2.45}$$

Substituting this expansion into (2.12), we obtain the **'cartesian' variant of the multipole expansion**, $\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \dots$, where

$$\phi^{(0)}(\mathbf{x}) = \frac{Q^{(0)}}{r}, \qquad \phi^{(1)}(\mathbf{x}) = Q^{(1)}_i\frac{x_i}{r^3}, \qquad \phi^{(2)}(\mathbf{x}) = \frac{Q^{(2)}_{ij}}{2}\frac{x_ix_j}{r^5}. \tag{2.46}$$

Here $Q^{(0)} = \int d^3x'\,\rho(\mathbf{x}')$ is the charge, $Q^{(1)}_i = \int d^3x'\,\rho(\mathbf{x}')x'_i$ are the components of the dipole moment, and $Q^{(2)}_{ij} = \int d^3x\,\rho(\mathbf{x}')(3x'_ix'_j - r'^2\delta_{ij})$ the cartesian variant of the quadrupole moments. It is a straightforward (yet somewhat tedious) exercise to work out the linear relation $Q^{(l)}_{ij} \leftrightarrow Q_{l,m}$ between the cartesian and the spherical multipole moments.

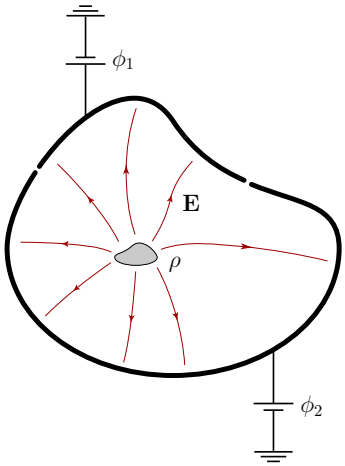
2.2.5 Boundary value problem II: finite space

Boundary conditions

Eq.(2.12) represents a solution to the Poisson equation with vanishing–potential boundary conditions at spatial infinity. In the text above we spoke of *the* solution, yet we haven't actually shown its uniqueness. A related and more general question: suppose we wish to compute the solution of

the Poisson equation in some bounded region of space B. Further suppose we wish to specify the functional profile of the potential at the boundaries ∂B of that volume. How would one do that? I.e. how much information do we *need* to specify to make the solution in B unique and when would the problem become over–specified. Further, given a complete specification of the boundary conditions, how does one solve the Poisson equation in practice? These are the questions we will address in this section.

▷ Info. The boundary value problem sketched above is not quite as academic as it might seem. For example, in electrical engineering, one often needs to compute the electric field created by charge distributions inside space domains limited by metallic boundaries (see the figure.)



Now, the potential inside an ideally conducting metallic surface must be constant. Reason: was there a potential gradient tangential to the surface, the corresponding electric field would make the mobile charges inside the metal re–arrange so as to eliminate that very potential drop. (Notice that the constancy of the potential implies that electric field lines are perpendicular to metallic surfaces.) However, by attaching electrodes to the boundary sheets, one can put these at any desired value of the potential (relative to the ground potential $\phi = 0$.) Further, the region of interest my by bounded by several, mutually isolated boundary regions, i.e. the boundary potential may be described by some piecewise constant function $\phi|_{\partial B}$. By taking the — gedanken — limit of a very large number of (mutually isolated) boundary sheets, we arrive at the prototypical boundary value problem specified by an arbitrary function $\phi|_{\partial B}$. Boundary conditions set in this way are denoted as **Dirichlet boundary conditions**. We will see below that the Dirichlet problem affords a unique solution (means extension of ϕ inside the domain.) That this must be so follows from our expectation that the setup outlined above is physically meaningful, i.e. will generate a unique electric field.

However, there are alternative ways of formulating a physically meaningful boundary problem: imagine a domain of space bounded by a layer of surface charges. In section 2.2.2 the field component normal to a surface charge layer, $\partial_n \phi$ is determined by the (local) amount of surface charge. This suggests that the specification of $\partial_n \phi$ everywhere at ∂B — so–called **Neumann boundary condtions** — also defines a uniquely solvable problem. However, the simultaneous specification of ϕ *and* its normal derivative (or any other additional information for that matter) over–specifies the problem.

We begin our analysis by proving the uniqueness of the solution to the Poisson equation with either Dirichlet or Neumann boundary conditions: Let the closed surface ∂B be the boundary of some region of space, B. (Parts, or even the whole of ∂B may be at spatial infinity.) On ∂B we specify either

▷ the potential $\phi|_{\partial B}$ (Dirichlet boundary conditions), or

▷ its normal derivative $\partial_n \phi|_{\partial B}$ (Neumann conditions.)

(Yet more generally, we may specify ϕ on a subset $S \subset \partial B$ and $\partial_n \phi$ on the complementary domain

$\partial B \setminus S$.) Then the solution to the Poisson equation is uniquely determined.

To prove this statement, let us assume that two solutions ϕ_1 and ϕ_2 obeying the same boundary conditions had been found. Applying Green's first identity (1.10) to the difference $\Gamma \equiv \phi_1 - \phi_2$, we obtain

$$\int_B d^3x \, (\Gamma \Delta \Gamma - \nabla\Gamma \cdot \nabla\Gamma) = \int_{\partial B} d\sigma \Gamma \partial_n \Gamma.$$

Using that $\Delta\Gamma = -\Delta\phi_1 + \Delta\phi_2 = 4\pi(\rho - \rho) = 0$ and that for all points in ∂B, either $\partial_n\Gamma = 0$ or $\Gamma = 0$, we conclude $\int d^3x' \, |\nabla\Gamma|^2 = 0$. Since $|\nabla\Gamma| \geq 0$, this relation implies $\nabla\Gamma = 0$ everywhere in B. Thus $\Gamma = \text{const.}$, i.e. the two potentials are equal up to an inessential constant (which for Dirichlet boundary conditions must be zero at the boundary, and hence everywhere.)

Formal solution of the boundary value problem

We next employ the method of Green functions to construct a formal solution of the boundary value problem. The attribute 'formal' signals that the construction will leave us with much of the hard work of obtaining a solution of the Poisson for concretely specified boundaries. What the method will yield, though, is a partial breakup of the problem into two prototype problems: (i) the solution of the Poisson equation for a δ–source and simplified boundary data, plus (ii) an integral.

Let the Green function $G(\mathbf{x}, \mathbf{x}')$ be a solution of the δ–source Poisson equation in $\mathbf{x}'$: $\Delta' G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$ (i.e. deviating from our earlier definition, we let the Laplace operator act on the coordinate $\mathbf{x}'$. However, this is just a matter of convention; G is still a Green function.) Now, a particular solution to this equation — viz. the one with vanishing boundary conditions at infinity — is $G_0(\mathbf{x}, \mathbf{x}') = (4\pi|\mathbf{x} - \mathbf{x}'|)^{-1}$. To implement other types of boundary conditions, we may use the freedom to add to any particular solution an arbitrary solution of the homogeneous equation (the Laplace equation), $F(\mathbf{x}, \mathbf{x}')$, where $\Delta' F(\mathbf{x}, \mathbf{x}') = 0$. Thus, the most general form of the Green function is given by

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}'). \tag{2.47}$$

Temporarily leaving the auxiliary function F unspecified, we next employ Greens' first identity (1.11) to relate the Green function to the potential ϕ. Choosing $\theta(\mathbf{x}') = G(\mathbf{x}, \mathbf{x}')$ and $\psi(\mathbf{x}') = \phi(\mathbf{x}')$, we then obtain the equation

$$\phi(\mathbf{x}) = 4\pi \int_B d^3x \, G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') + \int_{\partial B} d\sigma' \, \left[G(\mathbf{x}, \mathbf{x}') \partial_{n'} \phi(\mathbf{x}') - \phi(\mathbf{x}') \partial_{n'} G(\mathbf{x}, \mathbf{x}') \right]. \tag{2.48}$$

While for boundaries ∂B sent to infinity we are back to formula (2.12), for finite volumes, the surface terms on the r.h.s. implement the relevant boundary information. Specifically, for **Dirichlet boundary conditions**, we will try to determine a function F such that $G(\mathbf{x}, \mathbf{x}')|_{\mathbf{x}' \in \partial B} = 0$ obeys vanishing (Dirichlet) boundary conditions. Then,

$$\phi(\mathbf{x}) = 4\pi \int_B d^3x \, G(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') - \int_{\partial B} d\sigma' \, \phi(\mathbf{x}') \partial_{n'} G(\mathbf{x}, \mathbf{x}') \tag{2.49}$$

describes the potential in terms of the Green function and the pre–specified Dirichlet boundary data $\phi(\mathbf{x}' \in \partial B)$. Notice the logics of our strategy: by utilizing the Green function, we reduce the problem (general sources/general Dirichlet boundary data) to the much simpler one (δ–source/vanishing

Dirichlet boundary data). Once this latter problem has been solved, the solution of a general problem of the first kind can be obtained by integration.

Turning to the case of **Neumann boundary conditions** and following the same strategy as above, one might be tempted to set $\partial_{n'} G(\mathbf{x},\mathbf{x}')|_{\mathbf{x}'\in\partial B} = 0$. However, this requirement would be too strong: application of Stokes theorem obtains $\int_{\partial B} d\sigma'\, \partial_{n'} G(\mathbf{x},\mathbf{x}') = \int_B d^3x'\, \Delta' G(\mathbf{x},\mathbf{x}') = -1$, i.e. the normal derivative must not be globally vanishing. However, we may well set it to the constant value $\partial_{n'} G(\mathbf{x},\mathbf{x}')|_{\mathbf{x}'\in\partial B} = -A^{-1}$, where A is the area of ∂B.

Substituting this ansatz into (2.48), we obtain

$$\phi(\mathbf{x}) = 4\pi \int_B d^3x\, G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}') + \int_{\partial B} d\sigma'\, \partial_{n'}\phi(\mathbf{x}') G(\mathbf{x},\mathbf{x}') + \langle\phi\rangle_{\partial B}, \tag{2.50}$$

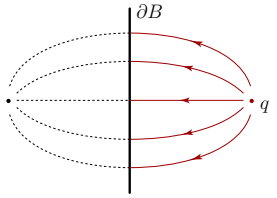
where the constant $\langle\phi\rangle_{\partial B} \equiv A^{-1}\int_{\partial B} d\sigma'\, \phi(\mathbf{x}')$ is the average value of the potential on the boundary. Since this constant will not affect the electric field, its presence is not of importance. (However, the need to solve (2.50) self consistently may create some headaches when it comes to actual computations.[14])

Before testing the method above on a concrete example, let us briefly discuss the **physical meaning of the function F in (2.47)**. Being a solution of the Laplace equation (inside B), we may think of the function F as the potential created by charges *outside* B. For any specific choice of the reference point $\mathbf{x}$ of the unit charge determining G, these outer charges modify the 'bare' potential $\sim |\mathbf{x}-\mathbf{x}'|^{-1}$ so as to conform with the boundary conditions. This way of thinking of the function F is not only of conceptual value; It provides one with a hint as to how to determine the function G in concrete applications. Especially in problems of a high degree of symmetry may try to introduce some 'image charges' outside B whose position/strength is chosen so as to adjust the boundary behaviour of ϕ appropriately.

Concrete solution of boundary value problems

First the bad news: there are no generally applicable solution strategies to obtain the Dirichlet or Neumann Green function; in most problems of practical interest one will need to resort to numerical procedures. However, in cases where one is dealing with a problem of exceptionally high symmetry, one may try to follow the logics outlined in the end of the previous section to manually construct a solution: Placing a system of fictitious charges — so called **image charges** — outside the reference volume B one may try to modify the potential inside B so as to conform with the boundary conditions. Let us introduce this method on two simple examples:

▷ Example: Consider a **charge distribution in front of an infinite conducting plane**. Choosing coordinates such that the plane coincides with the $x = 0$ plane and assuming that at spatial infinity it bends through the half space $x > 0$ into a closed surface ∂B we may apply the general formalism above. Consider, thus, a unit charge at some point $\mathbf{x}$ in front of the surface. We wish to compute the potential created by this charge (aka the Green function) where $\phi|_{\partial B} = 0$. To this end, we use the ansatz (2.47), where we think of the function F as the potential created by a mirror or image charge kept at the other side of the plane.



[14] Fortunately, in most Neumann–like problems, parts of the bounding surface are sent to infinity. In this case, $\langle\phi\rangle_{\partial B} = 0$ and the problem does not arise.

Symmetry considerations suggest[15] to choose the image charge to (a) be of equal and opposite strength -1, and (b) to place it at a coordinate vector $\mathbf{x}^s \equiv (-x, y, z)$ (i.e. the reflection of $\mathbf{x}$ at the grounded plane.) With this choice, the Green function assumes the form

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\left(\frac{1}{|\mathbf{x}' - \mathbf{x}|} - \frac{1}{|\mathbf{x}' + \mathbf{x}^s|}\right). \tag{2.51}$$

This function indeed solves the Poisson in the positive half plane (the interior of ∂B.) On the plane $x = 0$ it vanishes, i.e. it obeys vanishing Dirichlet boundary conditions as required. According to the general construction recipe, the potential of a general charge distribution in the positive half plane is given by

$$\phi(\mathbf{x}) \overset{x\geq 0}{=} 4\pi \int_{x>0} d^3x'\, G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}').$$

It is important to keep in mind that this result holds only in the positive half plane (the interior of B); the image charge is purely fictitious, i.e. there are no true charges in the negative half plane implying that $\phi(\mathbf{x})|_{x<0} = 0$.[16]

One may ask whether the second term (2.51) can be given a more 'physical' interpretation than that of the potential created by a phantom charge. Indeed, one should expect that a charge placed in front of a grounded conducting plane will lead to the generation of some surface charge (i.e. physically realized by electrons that are either attracted or repelled by the reference charge.) To compute the induced charge, we may employ Eq.(2.22), i.e. determine the jump in the normal component of the electric field across the surface. For the sake of simplicity, let us consider a point charge $q > 0$ sitting at $\mathbf{x} = (a, 0, 0)$ in front of the plane. According to Eq. (2.51), the potential created by this charge is given by

$$\phi(\mathbf{x}) = \begin{cases} q\left[((x-a)^2 + y^2 + z^2)^{-1/2} - ((x+a)^2 + y^2 + z^2)^{-1/2}\right], & x > 0, \\ 0, & x < 0. \end{cases}$$

Differentiating this result we obtain the surface charge density $\eta(y, z) = \Delta E_\perp/4\pi = -\partial_x\phi/4\pi = -(qa/2\pi)(a^2 + y^2 + z^2)^{-3/2}$. Notice that the (negative) induced charge is maximal at the point closest to the test charge and decays as $\sim r^{-3}$ for points in the plane far away from the point of closest proximity. The total amount of induced charge is given by

$$q_{\text{ind}} = \int_{\partial B} dydz\, \eta(y, z) = \frac{qa}{2\pi}\int_{\partial B} \frac{dydz}{(a^2 + y^2 + z^2)^{3/2}} = -q.$$

The totality of the induced charge completely 'screens' the test charge. Contrary to the fictitious image charge, the induced charge *is* physical (can be measured.). Also notice that the induced charge uniquely determines $\partial_n\phi$, i.e. it is surely impossible to freely impose both ϕ and $\partial_n\phi$.

▷ Example: As a second example, consider a charge distribution inside a conducting sphere of radius R (kept at potential $\phi = 0$.) Again, we attempt to construct the Green function of the problem by placing image charges into the region outside the system boundary. Let us *assume* that a single unit charge will be sufficient to equilibrate the potential on the sphere. By symmetry it is clear that this charge must be sit somewhere on the axis connecting the reference unit charge and the origin of the sphere.

Denoting the unit vector in the direction of the unit charge by $\mathbf{n}$ and strength and distance of the image charge by s and d, respectively, the trial Green function is given by

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\left(\frac{1}{|r\mathbf{n} - r'\mathbf{n}'|} - \frac{s}{|d\mathbf{n} - r'\mathbf{n}'|}\right),$$

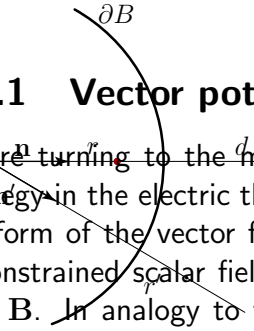
[15] As mentioned above, the method of image charges is applicable to problems with an exceptionally high degree of symmetry. Rather than 'computing' a system of suitable charges it is more customary to guess one.

[16] While the absence of charges merely implies $\phi(\mathbf{x})\big|_{x<0} = \text{const.}$, the vanishing of constant follows from the presence of a grounded ($\phi = 0$) boundary.

where $\mathbf{n}$ is the unit vector in direction $\mathbf{x}$. Setting $r' = R$ and reorganizing factors, this becomes

$$G(\mathbf{x}, \mathbf{x}')|_{r=R} = \frac{1}{4\pi}\left(\frac{1}{R|\mathbf{n}' - (r/R)\mathbf{n}|} - \frac{s}{d|\mathbf{n} - (R/d)\mathbf{n}'|}\right) \stackrel{!}{=} 0.$$

It is straightforward to verify that this expression vanishes if we set $d = R(R/r) > R$ and $s = R/r > 1$. The potential created by an arbitrary charge distribution inside the sphere is then given by (2.49). The analysis of the induced surface charge is left as an exercise.



2.3 Magnetostatics

2.3.1 Vector potential and solution of Maxwell equations

Before turning to the magnetic sector (2.2) of the static theory, let us recapitulate the solution strategy in the electric theory. Using the fact that $\nabla \times \mathbf{E} = 0$ (three scalar equations 'constraining' the form of the vector field $\mathbf{E}$), we wrote $\mathbf{E} = -\nabla\phi$, i.e. we obtained $\mathbf{E}$ as the derivative of an unconstrained scalar field. Similarly, in the magnetic case, $\nabla \cdot \mathbf{B} = 0$ constrains the form of the field $\mathbf{B}$. In analogy to the ansatz above we might, thus, attempt to represent $\mathbf{B}$ as $\mathbf{B} = D$(an unconstrained field), where D is a suitably chosen differential operator. Since, however, $\nabla \cdot \mathbf{B} = 0$ represents only one scalar constraint equation, the unconstrained auxiliary field will be of a more complex structure than the scalar potential ϕ. Indeed, we are going to show below that the equation $\nabla \cdot \mathbf{B} = 0$ implies that the magnetic field can be written as $\mathbf{B} = \nabla \times \mathbf{A}$, i.e. as the curl of an auxiliary field known as the (magnetic) **vector potential**. In magnetostatics, the vector potential plays a role analogous to the scalar potential of electrostatics.

▷ Math. To understand the mathematical principles behind the **existence of a vector potential**, consider the following problem (whose solution is of considerable importance in its own right): Let $g/\mathbf{f}$ be a scalar/source–free vector field ($\nabla \cdot \mathbf{f} = 0$). Is there a vector field $\mathbf{X}$ such that $g = \nabla \cdot \mathbf{X}$ are the sources of $\mathbf{X}$ and $\mathbf{f} = \nabla \times \mathbf{X}$ its circulation? If so, is $\mathbf{X}$ unique?

To answer these questions, let us make an ansatz $\mathbf{X} = \mathbf{X}_s + \mathbf{X}_c$, where

$$\begin{aligned} \nabla \times \mathbf{X}_s &= 0, & \nabla \cdot \mathbf{X}_s &= g, \\ \nabla \cdot \mathbf{X}_c &= 0, & \nabla \times \mathbf{X}_c &= \mathbf{f}, \end{aligned}$$

i.e. we decompose the sought–for vector field into a 'source component' $\mathbf{X}_s$ (curl–free but source–ful) and a complementary 'curl component' $\mathbf{X}_c$ (curl but no sources.)

We know that the circulation–less component $\mathbf{X}_s = -\nabla\phi$ can be written as the gradient of a scalar function.[17] This function must then be determined such that it satisfies the source condition, $\nabla \cdot \mathbf{X}_s = -\Delta\phi = g$. Turning to the complementary component $\mathbf{X}_c$, let us assume that the vanishing of the divergence can be resolved by making the ansatz, $\mathbf{X}_c = \nabla \times \mathbf{A}'$, i.e. we assume that the vanishing of the divergence implies the existence of a (vector) potential, very much like the vanishing of the curl implied the existence of a scalar potential. Surely this ansatz is consistent with $\nabla \cdot \mathbf{X}_c = 0$. To relate $\mathbf{A}'$ to the circulation of $\mathbf{X}_c$, we compute $\nabla \times \mathbf{X}_c = \nabla \times \nabla \times \mathbf{A}' = -\Delta\mathbf{A}' + \nabla(\nabla \cdot \mathbf{A}') \stackrel{!}{=} \mathbf{f}$. The problem posed by the solution of this equation can be simplified by noting that the addition of the gradient of an

[17] ..., where the minus sign has been introduced to stay close to the conventions of electrostatics.

arbitrary scalar function ψ to the vector potential, $\mathbf{A}' \to \mathbf{A} \equiv \mathbf{A}' + \nabla \cdot \psi$, does not change $\mathbf{X}_c$, i.e. the transformed vector potential is as good a vector potential as the original one. We now chose the function ψ such that $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}' + \Delta\psi = 0$, i.e. we use it to render $\mathbf{A}$ divergence-less. Our source equation then reduces to $-\Delta\mathbf{A} = \mathbf{f}$ or three independent Poisson equations $\Delta A_i = -f_i$ for the components of $\mathbf{A}$. These equations are solved (assuming infinite space boundary conditions for simplicity) by the formula $A_i(\mathbf{x}) = \int d^3x' \, |\mathbf{x} - \mathbf{x}'|^{-1} f_i(\mathbf{x}')$.

Expressed in terms of the scalar and the vector potential, $\mathbf{X} = -\nabla\phi + \nabla \times \mathbf{A}$. However, this choice of $\mathbf{X}$ is not unique: Without changing the circulation, we may add to $\mathbf{X}$ the gradient of a scalar function, $\mathbf{X} \to \mathbf{X} + \nabla h$. If h solves the Laplace equation, $\Delta h = 0$, the source content, too, remains unchanged. I.e. the vector field $\mathbf{X}$ is unique up to solutions of the Laplace equation.

The mathematical considerations outlined above solve much of the problem posed by the solution of Eqs.(2.2). The fact that the magnetic induction $\mathbf{B}$ is source free implies that it can be written as the curl of a vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$. The choice of the vector potential is not unique: without altering $\mathbf{B}$, we may add to $\mathbf{A}$ the gradient of an arbitrary function,

$$\mathbf{A} \to \mathbf{A} + \nabla f. \tag{2.52}$$

▷ Info. The transformation Eq.(2.52) is the most elementary example of a **gauge transformation**. Later on we will identify it as a specific case of the most general gauge transformation of electrodynamics. In fact, gauge transformations play a pivotal role in theories far more general than electrodynamics; they form the conceptual basis of various types of gauge theories. The list of important gauge theories includes quantum electrodynamics (QED), quantum chormodynamics (QCD), and many more.

The prototypical transformation (2.52) anticipates two crucial features of every gauge theory: (i) the freedom to choose amongst different realizations of an auxiliary field (Generally, fields possessing a gauge freedom are called **gauge fields**.) can be used to obtain a particularly simple, or problem adjusted representation of the theory. (ii) As the gauge transformation affects only the gauge field ($\mathbf{A}$) but not the derived 'physical field' ($\mathbf{B}$) all final results obtained by the theory must be independent of the choice of the gauge. This condition of **gauge invariance** imposes a strong consistency check on the validity of the theory and of physical approximation schemes.

Throughout it will be convenient to choose a gauge wherein $\nabla \cdot \mathbf{A} = 0$, the so–called **Coulomb gauge**. (Recall the logics of the discussion above: for an arbitrary configuration $\mathbf{A}'$, we may chose a gauge function f such that $\Delta f = -\nabla \cdot \mathbf{A}'$ solves the Poisson equation. Then, the gauged vector potential $\mathbf{A} \equiv \mathbf{A}' + \nabla f$ is source–free. Without loss of generality we may thus assume that $\mathbf{A}$ has been source–free from the outset.) Using the vacuum identification $\mathbf{H} = \mathbf{B}$ and substituting the ansatz $\mathbf{B} = \nabla \times \mathbf{A}$ into the Maxwell equation $\nabla \times \mathbf{H} = 4\pi c^{-1}\mathbf{j}$ we then obtain $\nabla \times (\nabla \times \mathbf{A}) = -\Delta\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\Delta\mathbf{A} = 4\pi c^{-1}\mathbf{j}$, i.e. the vector potential is obtained by solving the three independent Poisson equations $\Delta A_i = -4\pi c^{-1} j_i$. These equations are mathematically equivalent to their electrostatic analog $\Delta\phi = -4\pi\rho$ discussed above. Making the proper identifications and assuming infinite space boundary conditions we thus obtain the solution

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{1}{c} \int d^3x' \, \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.} \tag{2.53}$$

Acting on this equation with the differential operator $\nabla\times$ and using the auxiliary identity $\nabla \times (f(\mathbf{x})\mathbf{v}(\mathbf{x}')) = \nabla f(\mathbf{x}) \times \mathbf{v}(\mathbf{x}')$ we find that the magnetic field created by the current distribution is given by

$$\boxed{\mathbf{B}(\mathbf{x}) = \frac{1}{c}\int d^3x' \, \frac{\mathbf{j}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}.} \tag{2.54}$$

After the experimentalists who first quantitatively explored the connection between magnetic fields and currents, Eq.(2.54) is called the **law of Biot and Savart**.

Two more remarks on the result (2.53) are in place: Firstly, notice that the Eq.(2.53) was derived for vanishing boundary conditions at infinity, $A(\mathbf{x})|_{|\mathbf{x}|\to\infty} \to 0$. In principle, however, our complete discussion of Poisson equations subject to more complex boundary conditions in section 2.2.5 may be carried over to the magnetic case. Yet, unlike with electrostatics (where both Dirichlet and Neumann boundary conditions could be given a clear physical meaning) we do not yet know how to interpret 'magnetic boundary conditions' on the behaviour of $\mathbf{A}$. For a discussion of this point we refer to chapter 4 below. Secondly, we need to verify that the vector potential (2.53) indeed obeys our working assumption, i.e. the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. To this end, we note that the Fourier transform of the Poisson equation $\Delta\mathbf{A} = -\frac{4\pi}{c}\mathbf{j}$ is given by $-k^2\mathbf{A}(\mathbf{k}) = -\frac{4\pi}{c}\mathbf{j}(\mathbf{k})$. Solving for $\mathbf{A}(\mathbf{k})$ we obtain $\mathbf{A}(\mathbf{k}) = \frac{4\pi}{c}k^{-2}\mathbf{j}(\mathbf{k})$.[18] Taking the scalar product of this equation with the vector $\mathbf{k}$ and noting that $\nabla\cdot\mathbf{j}$ Fourier transforms to $\mathbf{k}\cdot\mathbf{j} = 0$, we obtain $\mathbf{k}\cdot\mathbf{A} = 0$ or $\nabla\cdot\mathbf{A} = 0$.

2.3.2 Magnetic multipole expansion

To better understand the physical meaning of Eqs.(2.53) and (2.54) above, let us consider a situation where the current flow in the system is confined to a small region B is space. We want to compute the magentic field created by the currents at points far away from B. As in our discussion in section 2.2.4, we might apply the addition theorem to expand the distance function in (2.53) in spherical harmonics, thus generating a magnetic variant of the multipole expansion.

▷ Info. Indeed, expansions of this type are of considerable applied relevance: for example, the construction of magnetic traps is based on **quadrupole magnets**, i.e. magnets whose multipole expansion starts at sub–leading order. Similarly, the particle beams in accelerators are commonly focused by sequences of quadrupole magnets (occasionally even octupole magnets or magnets of yet higher order.)

However, presently, we shall restrict ourselves to a simpler 'brute force' Taylor expansion of the distance–denominator, similar to the one outlined in the end of section 2.2.4. To present the results of this expansion in a convenient manner, we will make use of the auxiliary identity

$$\int d^3x (f\mathbf{j}\cdot\nabla g + g\mathbf{j}\cdot\nabla f) = 0, \tag{2.55}$$

which holds for arbitrary functions f, g and source–free vector fields $\mathbf{j}$. This equation is proven by a straightforward integration by parts. Application of (2.55) to the cases $g = x_i$, $f = 1$ and $f = x_i$,

[18] To establish the equivalence between the Fourier representation and the real space representation above, recall the remarks made in the paragraph below Eq. (2.17).

$g = x_j$ obtains the two relations

$$\int d^3x\, j_i = 0, \qquad \int d^3x'(x_i j_j + j_j x_i) = 0. \tag{2.56}$$

Substituting the expansion (2.45) into (2.53), we obtain the series $\mathbf{A} = \mathbf{A}^{(0)} + \mathbf{A}^{(1)} + \ldots$, which should be compared to the scalar potential series (2.46). Specifically, the analog of the electrostatic monopole term is given by

$$\mathbf{A}^{(0)} = \frac{1}{cr}\int d^3x'\,\mathbf{j}(\mathbf{x}') = 0,$$

where we used the first of the two relations (2.56). The vanishing of the first moment reflects the **absence of magnetic monopoles**.

Turning to the second moment, we obtain

$$A_i^{(1)} = \frac{x_j}{cr^3}\int d^3x'\, x'_j j_i \overset{(2.56)}{=} \frac{x_j}{2cr^3}\int d^3x'\,(x'_j j_i - x'_i j_j) = \frac{x_j \epsilon_{kji}\epsilon_{kj'i'}}{2cr^3}\int d^3x'\, x'_{j'} j_{i'} =$$
$$= \frac{x_j \epsilon_{kji}}{2cr^3}\int d^3x'\,(\mathbf{x}'\times\mathbf{j})_k = -\frac{1}{2cr^3}\left[\mathbf{x}\times\int d^3x'\,\mathbf{x}'\times\mathbf{j}\right]_i.$$

In close analogy to the electric dipole moment, we define the **magnetic dipole moment**

$$\mathbf{m} \equiv \frac{1}{2c}\int d^3x\,\mathbf{x}\times\mathbf{j}. \tag{2.57}$$

Expressed in terms of this quantity, the dipolar contribution to the vector potential is given by

$$\boxed{\mathbf{A}^{(1)} = \frac{\mathbf{m}\times\mathbf{x}}{r^3}.} \tag{2.58}$$

In principle, the series expansion of the vector potential may now be driven to higher (quadrupolar) orders. However, for the sake of simplicity, we will restrict our discussion to the discussion of the dominant contribution to the series, Eq.(2.58).

Taking the curl of (2.58), and noting that $\nabla\times(\mathbf{x}\times\mathbf{v}) = -(\nabla\cdot\mathbf{x})\mathbf{v} = -3\mathbf{v}$ we obtain the magnetic field created by the current dipole,

$$\mathbf{B}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{r^3}, \tag{2.59}$$

where $\mathbf{n}$ is the unit vector in $\mathbf{x}$–direction. Notice the structural equivalence of this expression to its electric analog, Eq.(2.16).

Before proceeding, let us briefly discuss the magnetic dipole moments of two particular types current of distributions of considerable practical relevance:

- ▷ Often, the current sources are realized as (approximately) **closed current loops**, each carrying a constant current I. In this case (cf. Eq.(1.22)) the volume integral (2.57) reduces to

$$\mathbf{m} = -\frac{I}{2c}\oint_\gamma d\mathbf{s}\times\mathbf{x}. \tag{2.60}$$

Particularly simple expressions are obtained for planar current loops, i.e. in cases where the curve γ lies in a plane. Introducing coordinates such that the unit vector normal to the plane $\mathbf{n} = \mathbf{e}_3$ coincides with the coordinate vector $\mathbf{e}_3$, we then obtain for the ith component of the magnetic moment

$$m_i = -\frac{I}{2c}\mathbf{e}_i \cdot \oint_\gamma d\mathbf{s} \times \mathbf{x} = -\frac{I}{2c}\oint_\gamma \mathbf{e}_i \cdot (d\mathbf{s} \times \mathbf{x}) = -\frac{I}{2c}\oint_\gamma d\mathbf{s} \cdot (\mathbf{x} \times d\mathbf{e}_i).$$

Application of Stokes law then yields $m_i = -\frac{I}{2c}\int_{S(\gamma)} \int d\sigma\, \mathbf{e}_3 \cdot (\nabla \times (\mathbf{x} \times \mathbf{e}_i))$. Noting that $\mathbf{e}_3 \cdot (\nabla \times (\mathbf{x} \times \mathbf{e}_i)) = -2\delta_{3i}$, we obtain

$$\mathbf{m} = \frac{AI}{c}\mathbf{e}_3,$$

i.e. a magnetic moment perpendicular to the plane supporting the current loop and proportional to the area of the loop.

▷ Imagine a **system of point particles** at coordinates $\mathbf{x}_i$ where each particle carries charge q_i, is of mass m_i, and moves with velocity $\mathbf{v}_i$. In this case, $\mathbf{j} = \sum_i \mathbf{v}_i q_i \delta(\mathbf{x} - \mathbf{x}_i)$. Equation (2.57) reduces to

$$\frac{1}{2c}\sum_i q_i \mathbf{x}_i \times \mathbf{v}_i = \frac{1}{2c}\sum_i \frac{q_i}{m_i}\mathbf{l}_i, \tag{2.61}$$

where $\mathbf{l} = \mathbf{x} \times (m\mathbf{v})$ is the angular momentum carried by a point particle. [Only parts of the magnetic moment carried by genuine elementary particles are due to their orbital angular momentum $\mathbf{l}$. A second contribution stems from their 'intrinsic' angular momentum or **spin**. Specifically, for an electron at rest, $\mathbf{m} = 2.002 \times \frac{e}{2cm}\mathbf{S}$, where $|\mathbf{S}| = 1/2$ is the electron spin and the pre–factor 2.002 is known as the **g–factor** of the electron.]

2.3.3 Magnetic forces

To understand the physical meaning of the connections (currents $\rightsquigarrow$ fields) derived above, let us explore an intuitively accessible quantity, viz. the mechanical forces created by a current distribution. Assuming that the charge density of the mobile charge carriers in the wire is given by ρ, and that these charges move with a velocity $\mathbf{v}$, the current density in the wire is given by $\mathbf{j} = \rho\mathbf{v}$.[19] Comparison with the Lorentz force law (1.6) shows that in a magnetic field the wire will be subject to a force density $= \mathbf{c^{-1}}\rho\mathbf{v} \times \mathbf{B}$. Specifically, for a thin wire of cross section dA, and carrying a current $I = \rho dAv$ (v is the component of $\mathbf{v} = v\mathbf{e}_\parallel$ along the wire.) the force acting on a segment of length dl will be $d\mathbf{F} = \mathbf{f}dAdl = c^{-1}\rho dAdl(v\mathbf{e}_\parallel) \times \mathbf{B} = c^{-1}Id\mathbf{s} \times d\mathbf{B}$. Summarizing, the force acting on a line element $d\mathbf{s}$ of a wire carrying a current I is given by

$$\boxed{d\mathbf{f} = \frac{I}{c}d\mathbf{s} \times \mathbf{B}.} \tag{2.62}$$

In the following, we discuss a few applications of the prototypical force–formula (2.62):

[19] The amount of charge flowing through a small surface element $d\sigma$ during a time interval dt is $dQ = d\sigma\rho\mathbf{n} \cdot \mathbf{v}dt$. Dividing by dt we find that the current through $d\sigma$ is given by $I_{d\sigma} = d\sigma\mathbf{n} \cdot (\rho\mathbf{v})$ which means that $\mathbf{j} = \rho\mathbf{v}$ is the current density.

Forces on current loops

Consider a **single closed current loop** γ carrying a current I. The current flow will generate a magnetic field (2.54) which will in turn act on the line elements of the loop, as described by (2.62). One may thus wonder whether the loop exerts a net force on itself. Integrating over (2.62), we find that this force is given by

$$\begin{aligned}\mathbf{F} = \oint_\gamma d\mathbf{f} = \frac{I}{c}\oint_\gamma d\mathbf{s}\times\mathbf{B} &\overset{(2.54)}{=} \frac{I^2}{c^2}\oint_\gamma\oint_\gamma \frac{d\mathbf{s}\times(d\mathbf{s}'\times(\mathbf{x}-\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|^3} = \\ &= \frac{I^2}{c^2}\oint_\gamma\oint_\gamma\left(\frac{d\mathbf{s}'(d\mathbf{s}\cdot(\mathbf{x}-\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|^3} - \frac{(\mathbf{x}-\mathbf{x}')d\mathbf{s}\cdot d\mathbf{s}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) = \\ &= -\frac{I^2}{c^2}\oint_\gamma d\mathbf{s}\cdot\oint_\gamma d\mathbf{s}'\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 0.\end{aligned}$$

The second term in the second line vanishes because it changes sign under coordinate interchange $\mathbf{x}\leftrightarrow\mathbf{x}'$. In the third line we used that $(\mathbf{x}-\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{-3} = -\nabla|\mathbf{x}-\mathbf{x}'|^{-1}$ and that the integral of the gradient of a function over a closed loop vanishes. We conclude that *a closed current loop does not exert a net force onto itself.*

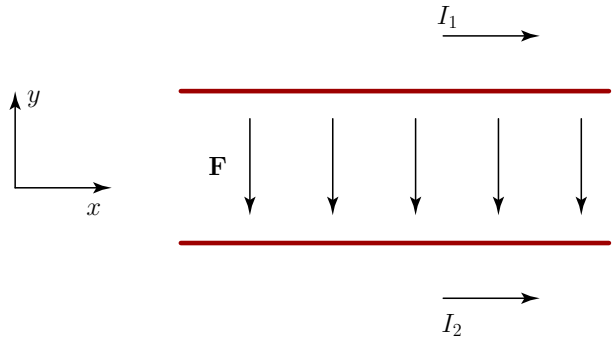


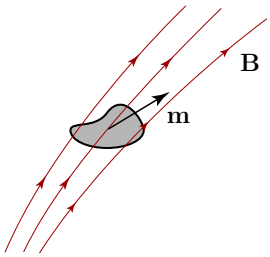
Figure 2.2: Two infinite wires exerting a force on each other

However, distinct current distributions do, in general, act by magnetic forces onto each other. By way of example, consider **two infinite parallel wires** kept at a distance d and carrying currents I_1 and I_2, resp. Choosing coordinates as shown in the figure and parameterizing the line elements as $d\mathbf{s}_{1,2} = dx_{1,2}\,\mathbf{e}_x$, we obtain the force $\mathbf{F}_{12}$ wire #2 exerts on wire #1 as

$$\begin{aligned}\mathbf{F}_{12} &= \frac{I_1 I_2}{c^2}\int dx_1\int dx_2\frac{\mathbf{e}_x\times\mathbf{e}_x\times(\mathbf{e}_x(x_1-x_2)-\mathbf{e}_y d)}{|(x_1-x_2)^2+d^2|^3}) = \\ &= \mathbf{e}_y\frac{I_1 I_2 d}{c^2}\int dx_1\int dx_2\frac{1}{(x_2^2+d^2)^{3/2}} = \mathbf{e}_y\frac{2I_1I_2}{c^2 d}\int dx_1,\end{aligned}$$

where in the second line we used $\int_{-\infty}^{\infty} dx(x^2+d^2)^{-3/2} = 2/d^2$. Reflecting their infinite length, the wires exert an infinite force on each other. However, the force per unit length $\frac{d\mathbf{F}}{dx} = \frac{2I_1I_2}{c^2 d}\mathbf{e}_y$ is finite and depends inverse quadratically on the distance between the wires. The force is attractive/repulsive for currents flowing in the same/opposite direction. (One of the more prominent manifestations of the attractive magnetic forces between parallel current flows is the phenomenon of current implosion in hollow conductors.)

Forces on local current distributions



Consider a spatially localized current distribution in an external magnetic field. We wish to compute **total force** acting on the distribution

$$\mathbf{F} = \int d^3x\, \mathbf{f} = \frac{1}{c}\int d^3x\, \mathbf{j}\times\mathbf{B}. \tag{2.63}$$

Choosing the origin of the coordinate system somewhere inside the current distribution, and assuming that the magnetic field varies slowly across the extent of the current flow, one may Taylor expand $\mathbf{B}$ in the Lorentz force formula:

$$\mathbf{F} = \frac{1}{c}\int d^3x\, \mathbf{j}(\mathbf{x})\times\left[\mathbf{B}(0) + \mathbf{x}\cdot\nabla\mathbf{B}(0) + \ldots\right].$$

($\mathbf{x}\cdot\nabla\mathbf{B}$ is a vector with components $x_i\partial_i B_j$.) The auxiliary identity (2.56) implies that the first contribution to the expansion vanish. The ith component of the force is thus given by

$$\begin{aligned} F_i &\simeq \frac{1}{c}\int d^3x\, \epsilon_{ijk}\, j_j\, x_l\, \partial_l\, B_k \overset{(2.56)}{=} \frac{1}{2c}\int d^3x\, \epsilon_{ijk}\epsilon_{mjl}(\mathbf{j}\times\mathbf{x})_m\partial_l B_k = \\ &= \frac{1}{2c}\int d^3x\, (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(\mathbf{j}\times\mathbf{x})_m\partial_l B_k = \frac{1}{2c}\int d^3x\, ((\mathbf{j}\times\mathbf{x})_i\partial_k B_k - (\mathbf{j}\times\mathbf{x})_k\partial_i B_k) = \\ &= \partial_i|_{\mathbf{x}=0}(\mathbf{m}\cdot\mathbf{B}(\mathbf{x})), \end{aligned}$$

or

$$\boxed{\mathbf{F} = \nabla|_{\mathbf{x}=0}(\mathbf{m}\cdot\mathbf{B}(\mathbf{x})),} \tag{2.64}$$

where in the second line we used $\nabla\cdot\mathbf{B} = \partial_l B_l = 0$ Thus, (a) the force is strongest if the magnetic moment is aligned with the magnetic field, and (b) proportional to the rate at which the external magnetic field varies. Specifically, for a uniform field, no forces act.

While the force on a magnetic moment relies on the presence of a field gradient, even a constant field does exert a finite **torque**

$$\mathbf{N} = \int d^3x\, \mathbf{x}\times\mathbf{f} = \frac{1}{c}\int d^3x\, \mathbf{x}\times(\mathbf{j}\times\mathbf{B}). \tag{2.65}$$

Approximating the field by its value at the origin,

$$\begin{aligned} N_i &= \frac{1}{c}\int d^3x\, \epsilon_{ijk}\epsilon_{klm}\, x_j j_l B(0)_m = \frac{1}{c}\int d^3x\, (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\, x_j j_l B(0)_m = \\ &= \frac{1}{c}\int d^3x\, \left(x_j j_i B(0)_j - x_j j_j B(0)_i\right) \overset{(2.56)}{=} \frac{1}{2c}\int d^3x\, \epsilon_{lji}(\mathbf{x}\times\mathbf{j})_l B(0)_j = \\ &= (\mathbf{m}\times\mathbf{B}(0))_i. \end{aligned}$$

Thus, the torque acting on the moment,

$$\boxed{\mathbf{N} = \mathbf{m}\times\mathbf{B}(0),} \tag{2.66}$$

is perpendicular to both the external field and the moment vector. It acts so as to align the field and the moment.

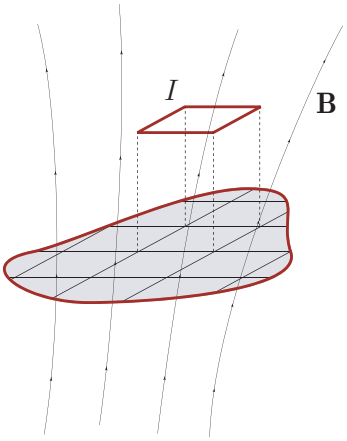
Chapter 3

Electro*dynamics*

3.1 Magnetic field energy

As a prelude to our discussion of the full set of Maxwell equations, let us address a question which, in principle, should have been answered in the previous chapter: What is the energy stored in a static magnetic field? In section 2.2.2, the analogous question for the electric field was answered in a constructive manner: we computed the *mechanical* energy required to build up a system of charges. It turned out that the answer could be formulated entirely in terms of the electric field, without explicit reference to the charge distribution creating it. By symmetry, one might expect a similar prescription to work in the magnetic case. Here, one would ask for the energy needed to build up a current distribution against the magnetic field created by those elements of the current distribution that have already been put in place. A moments thought, however, shows that this strategy is not quite as straightforwardly implemented as in the electric case: no matter how slowly we move a 'current loop' in an magnetic field, an electric field acting on the charge carriers maintaining the current in the loop will be induced — the induction law. Work will have to be done to maintain a constant current and it is this work function that essentially enters the energy balance of the current distribution.

To make this picture quantitative, consider a current loop carrying a current I. We may think of this loop as being consecutively built up by importing small current loops (carrying current I) from infinity (see the figure.) The currents flowing along adjacent segments of these loops will eventually cancel so that only the net current I flowing around the boundary remains. Let us, then, compute the work that needs to be done to bring in one of these loops from infinity.



Consider, first, an ordinary point particle kept at a (mechanical) potential U. The rate at which this potential changes if the particle changes its position is $d_t U = d_t U(\mathbf{x}(t)) = \nabla U \cdot d_t \mathbf{x} = -\mathbf{F}\cdot\mathbf{v}$, where $\mathbf{F}$ is the force acting on the particle. Specifically, for the charged particles moving in our prototypical current loops, $\mathbf{F} = q\mathbf{E}$, where $\mathbf{E}$ is the electric field induced by variation of the magnetic flux through the loop as it approaches from infinity.

Next consider an infinitesimal volume element d^3x inside the loop. Assuming that the charge carriers move at a velocity $\mathbf{v}$, the charge of the volume element is given by $q = \rho d^3x$ and rate of its potential change by $d_t\phi = -\rho d^3x \mathbf{v}\cdot\mathbf{E} = -d^3x\mathbf{j}\cdot\mathbf{E}$. We may now use the induction law to express the electric field in terms of magnetic quantities: $\int d\mathbf{s}\cdot\mathbf{E} = -c^{-1}\int_{\delta S} d\sigma\,\mathbf{n}\cdot d_t\mathbf{B} = -c^{-1}d_t\int_{\delta S} d\sigma\,\mathbf{n}\cdot d_t(\nabla\times\mathbf{A}) = -c^{-1}\int d\mathbf{s}\cdot d_t\mathbf{A}$. Since this holds regardless of the geometry of the loop, we have $\mathbf{E} = -c^{-1}d_t\mathbf{A}$, where $d_t\mathbf{A}$ is the change in vector potential due to the movement of the loop. Thus, the rate at which the potential of a volume element changes is given by $d_t\phi = c^{-1}d^3x\mathbf{j}\cdot d_t\mathbf{A}$. Integrating over time and space, we find that the total potential energy of the loop due to the presence of a vector potential is given by

$$E = \frac{1}{c}\int d^3x\,\mathbf{j}\cdot\mathbf{A}. \tag{3.1}$$

Although derived for the specific case of a current loop, Eq.(3.1) holds for general current distributions subject to a magnetic field. (For example, for the current density carried by a point particle at $\mathbf{x}(t)$, $\mathbf{j} = q\delta(\mathbf{x}-\mathbf{x}(t))\mathbf{v}(t)$, we obtain $E = q\mathbf{v}(t)\cdot\mathbf{A}(\mathbf{x}(t))$, i.e. the familiar Lorentz–force contribution to the Lagrangian of a charged particle.)

Now, assume that we shift the loop at *fixed current* against the magnetic field. The change in potential energy corresponding to a small shift is given by $\delta E = c^{-1}\int d^3x\,\mathbf{j}\cdot\delta\mathbf{A}$, where $\nabla\times\delta\mathbf{A} = \delta\mathbf{B}$ denotes the change in the field strength. Using that $\nabla\times\mathbf{H} = 4\pi c^{-1}\mathbf{j}$, we represent δE as

$$\begin{aligned}\delta E = \frac{1}{4\pi}\int d^3x\,(\nabla\times\mathbf{H})\cdot\delta\mathbf{A} = \frac{1}{4\pi}\int d^3x\,\epsilon_{ijk}(\partial_j H_k)\delta A_i = \\ = -\frac{1}{4\pi}\int d^3x\,H_k\epsilon_{ijk}\partial_j\delta A_i = \frac{1}{4\pi}\int d^3x\,\mathbf{H}\cdot\delta\mathbf{B},\end{aligned}$$

where in the integration by parts we noted that due to the spatial decay of the fields no surface terms at infinitey arise. Due to the linear relation $\mathbf{H} = \mu_0^{-1}\mathbf{B}$, we may write $\mathbf{H}\cdot\delta\mathbf{B} = \delta(\mathbf{H}\cdot\mathbf{B})/2$, i.e. $\delta E = \frac{1}{8\pi}\delta\int\mathbf{B}\cdot\mathbf{E}$. Finally, summing over all shifts required to bring the current loop in from infinity, we obtain

$$\boxed{E = \frac{1}{8\pi}\int d^3x\,\mathbf{H}\cdot\mathbf{B}} \tag{3.2}$$

for the **magnetic field energy**. Notice (a) that we have again managed to express the energy of the system entirely in terms of the fields, i.e. without explicit reference to the sources creating these fields and (b) the structural similarity to the electric field energy (2.20).

3.2 Electromagnetic gauge field

Consider the full set of Maxwell equations in vacuum ($\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$),

$$\begin{aligned}\nabla\cdot\mathbf{E} &= 4\pi\rho,\\ \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} &= \frac{4\pi}{c}\mathbf{j},\\ \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} &= 0,\\ \nabla\cdot\mathbf{B} &= 0.\end{aligned} \tag{3.3}$$

As in previous sections we will try to use constraints inherent to these equations to compactify them to a smaller set of equations. As in section 2.3, the equation $\nabla \cdot \mathbf{B} = 0$ implies

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.} \tag{3.4}$$

(However, we may no longer expect $\mathbf{A}$ to be time independent.) Now, substitute this representation into the law of induction: $\nabla \times (\mathbf{E} + c^{-1}\partial_t \mathbf{A}) = 0$. This implies that $\mathbf{E} + c^{-1}\partial_t \mathbf{A} = -\nabla\phi$ can be written as the gradient of a scalar potential, or

$$\boxed{\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t \mathbf{A}.} \tag{3.5}$$

We have, thus, managed to represent the electromagnetic fields as in terms of derivatives of a generalized four–component potential $A = \{A_\mu\} = (\phi, -\mathbf{A})$. (The negative sign multiplying $\mathbf{A}$ has been introduced for later reference.) Substituting Eqs. (3.4) and (3.5) into the inhomogeneous Maxwell equations, we obtain

$$\begin{aligned} -\Delta\phi - \tfrac{1}{c}\partial_t \nabla \cdot \mathbf{A} &= 4\pi\rho, \\ -\Delta\mathbf{A} + \tfrac{1}{c^2}\partial_t^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) + \tfrac{1}{c}\partial_t \nabla\phi &= \tfrac{4\pi}{c}\mathbf{j} \end{aligned} \tag{3.6}$$

These equations do not look particularly inviting. However, as in section 2.3 we may observe that the choice of the generalized vector potential $\mathbf{A}$ is not unique; this freedom can be used to transform Eq.(3.6) into a more manageable form: For an arbitrary function $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, (\mathbf{x}, t) \mapsto f(\mathbf{x}, t)$. The transformation $\mathbf{A} \to \mathbf{A} + \nabla f$ leaves the magnetic field unchanged while the electric field changes according to $\mathbf{E} \to \mathbf{E} - c^{-1}\partial_t \nabla f$. If, however, we synchronously redefine the scalar potential as $\phi \to \phi - c^{-1}\partial_t f$, the electric field, too, will not be affected by the transformation. Summarizing, the **generalized gauge transformation**

$$\begin{aligned} \mathbf{A} &\to \mathbf{A} + \nabla f, \\ \phi &\to \phi - \frac{1}{c}\partial_t f, \end{aligned} \tag{3.7}$$

leaves the electromagnetic fields (3.4) and (3.5) unchanged. (In the four component shorthand notation introduced above, the gauge transformation assumes easy–to–memorize the form $A_\mu \to A_\mu - \partial_\mu f$, where the four–derivative operator $\{\partial_\mu\} = (\partial_0, \nabla)$ and $x_0 = ct$ as before.)

The gauge freedom may be used to transform the vector potential into one of several convenient representations. Of particular relevance to the solution of the time dependent Maxwell equations is the so–called **Lorentz gauge**

$$\boxed{\nabla \cdot \mathbf{A} + \frac{1}{c}\partial_t \phi = 0.} \tag{3.8}$$

This equation, too, affords a compact four–vector notation: for a general vector $\{v_\mu\} = (v_0, \mathbf{v})$ we define a vector $\{v^\mu\} \equiv (v_0, -\mathbf{v})$, i.e. a an object with 'raised components' that differs from the one with 'lowered components' by a sign change in the space–like sector. Using this notation, the Lorentz condition assumes the form $\partial_\mu A^\mu = 0$. It is always possible to satisfy this condition by a suitable gauge transformation. Indeed, if A' does not obey the Lorentz condition, we may define $A_\mu = A'_\mu - \partial_\mu f$ to obtain $0 \stackrel{!}{=} \partial_\mu A^\mu = \partial_\mu A'^\mu - \partial_\mu \partial^\mu f$. If we chose f so as to satisfy the equation

$\partial_\mu f^\mu = \partial_\mu A'^\mu$, the Lorentz equation is satisfied. Expressed in terms of space and time components, this latter equation assumes the form

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right) f = -\left(\nabla \cdot \mathbf{A}' + \frac{1}{c}\partial_t \phi'\right),$$

i.e. a wave equation with inhomogeneity $-(\nabla \cdot \mathbf{A} + c^{-1}\partial_t \phi)$. We shall see momentarily that such equations can always be solved, i.e. an implementation of the Lorentz gauge condition is possible.

In the Lorentz gauge, the Maxwell equations assume the simplified form

$$\begin{aligned}\left(\Delta - \frac{1}{c^2}\partial_t^2\right)\phi &= -4\pi\rho, \\ \left(\Delta - \frac{1}{c^2}\partial_t^2\right)\mathbf{A} &= -\frac{4\pi}{c}\mathbf{j}.\end{aligned} \tag{3.9}$$

In combination with the gauge condition (3.8), Eqs. (3.9) are fully equivalent to the set of Maxwell equations (3.3). Again, the four–vector notation may be employed to compactify the notation still further. With $\partial_\mu \partial^\mu = -\Delta + c^{-2}\partial_t^2$ and $j_\mu = (c\rho, -\mathbf{j})$, the potential equations assume the form

$$\boxed{\partial_\nu \partial^\nu A_\mu = \frac{4\pi}{c} j_\mu, \qquad \partial_\mu A^\mu = 0.} \tag{3.10}$$

Before turning to the discussion of the solution of these equations a few general remarks are in order:

- ▷ The Lorentz condition is the prevalent gauge choice in electrodynamics because (a) it brings the Maxwell equations into a maximally simple form and (b) will turn out below to be invariant under the most general class of coordinate transformations, the Lorentz transformations see below. (It is worthwhile to note that the Lorentz condition does not unambiguously fix the gauge of the vector potential. Indeed, we may add to any Lorentz gauge vector potential A_μ a function $A_\mu \to A_\mu + f_\mu$ satisfying the homogeneous wave equation $\partial^\mu \partial_\mu f = 0$ without altering the condition $\partial^\mu A_\mu = 0$.) Other gauge conditions frequently employed include
- ▷ the **Coulomb gauge** or **radiation gauge** $\nabla \cdot \mathbf{A} = 0$ (employed earlier in section 2.3.) The advantage of this gauge is that the scalar Maxwell equation assumes the simple form of a Poisson equation $\Delta\phi(\mathbf{x}, t) = 4\pi\rho(\mathbf{x}, t)$ which is solved by $\phi(\mathbf{x}, t) = \int d^3x' \rho(\mathbf{x}, t)/|\mathbf{x} - \mathbf{x}'|$, i.e. by an instantaneous 'Coulomb potential' (hence the name Coulomb gauge.) This gauge representation has also proven advantageous within the context of quantum electrodynamics. However, we won't discuss it any further in this text.

3.3 Electromagnetic waves in vacuum

As a warmup to our discussion of the full problem posed by the solution of Eqs. (3.9), we consider the vacuum problem, i.e. a situation where no sources are present, $j_\mu = 0$. Taking the curl of the second of Eqs. (3.9) and using (3.4) we then find

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right)\mathbf{B} = 0.$$

Similarly, adding to the gradient of the first equation c^{-1} times the time derivative of the second, and using Eq.(3.5), we obtain

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right)\mathbf{E} = 0,$$

i.e. in vacuum both the electric field and the magnetic field obey homogeneous wave equations.

3.3.1 Solution of the homogeneous wave equations

The homogeneous wave equations are conveniently solved by Fourier transformation. To this end, we define a four–dimensional variant of Eq. (2.10),

$$\begin{aligned}\tilde{f}(\omega,\mathbf{k}) &= \frac{1}{(2\pi)^4}\int d^3x dt\, f(t,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}, \\ f(t,\mathbf{x}) &= \int d^3k d\omega\, f(\omega,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t},\end{aligned} \tag{3.11}$$

The one difference to Eq.(2.10) is that the sign–convention in the exponent of the (ω/t)–sector of the transform differs from that in the $(\mathbf{k},\mathbf{x})$–sector. We next subject the homogeneous wave equation

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right)\psi(t,\mathbf{x}) = 0,$$

(where ψ is meant to represent any of the components of the $\mathbf{E}$– or $\mathbf{B}$–field) to this transformation and obtain

$$\left(k^2 - \left(\frac{\omega}{c}\right)^2\right)\psi(\omega,\mathbf{k}) = 0.$$

Evidently, the solution ψ must vanish for all values $(\omega,\mathbf{k})$, except for those for which the factor $k^2-(\omega/c)^2=0$. We may thus write

$$\psi(\omega,\mathbf{k}) = c_+(\mathbf{k})\delta(\omega-kc) + c_-(\mathbf{k})\delta(\omega+kc),$$

where $c_\pm \in \mathbb{C}$ are arbitrary complex functions of the **wave vector** $\mathbf{k}$. Substituting this representation into the inverse transform, we obtain the general **solution of the scalar homogeneous wave equation**

$$\psi(t,\mathbf{x}) = \int d^3k \left(c_+(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-ckt)} + c_-(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}+ckt)}\right). \tag{3.12}$$

The general solution is obtained by linear superposition of elementary **plane waves**, $e^{i(\mathbf{k}\cdot\mathbf{x}\mp ckt)}$, where each wave is weighted with an arbitrary coefficient $c_\pm(\mathbf{k})$. The elementary constituents are called waves because for any fixed instance of space, $\mathbf{x}$/time, t they harmonically depend on the complementary argument time, t/position vector, $\mathbf{x}$. The waves are planar in the sense that for all points in the plane fixed by the condition $\mathbf{x}\cdot\mathbf{k} = \text{const.}$ the phase of the wave is identical, i.e. the set of points $\mathbf{k}\cdot\mathbf{x} = \text{const.}$ defines a 'wave front' perpendicular to the wave vector $\mathbf{k}$. The spacing between consequtive wave fronts with the same phase $\arg(\exp(i(\mathbf{k}\cdot\mathbf{x}-ckt))$ is given by $\frac{\Delta x = 2\pi}{k\equiv\lambda}$, where λ is the **wave length** of the wave and $\lambda^{-1} = 2\pi/k$ its **wave number**. The temporal oscillation period of the wave fronts is set by $2\pi/ck$.

Focusing on a fixed wave vector $\mathbf{k}$, we next generalize our results to the vectorial problem posed by the homogeneous wave equations. Since every component of the fields $\mathbf{E}$ and $\mathbf{B}$ is subject to its own independent wave equation, we may write down the prototypical solutions

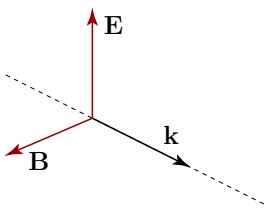
$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \qquad \mathbf{B}(\mathbf{x},t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{3.13}$$

where we introduced the abbreviation $\omega = ck$ and $\mathbf{E}_0, \mathbf{B}_0 \in \mathbb{C}^3$ are constant coefficient vectors. The Maxwell equations $\nabla\cdot\mathbf{B} = 0$ and (vacuum) $\nabla\cdot\mathbf{E} = 0$ imply the condition $\mathbf{k}\cdot\mathbf{E}_0 = \mathbf{k}\cdot\mathbf{B}_0 = 0$, i.e. the coefficient vectors are orthogonal to the wave vector. Waves of this type, oscillating in a direction perpendicular to the wave vector, are called **transverse waves**. Finally, evaluating Eqs.(3.13) on the law of induction $\nabla\times\mathbf{E} + c^{-1}\partial_t\mathbf{B} = 0$, we obtain the additional equation $\mathbf{B}_0 = \mathbf{n}_\mathbf{k}\times\mathbf{E}_0$, i.e. the vector $\mathbf{B}_0$ is perpendicular to both $\mathbf{k}$ and $\mathbf{E}_0$, and of equal magnitude as $\mathbf{E}_0$. Summarizing, the vectors

$$\mathbf{k}\perp\mathbf{E}\perp\mathbf{B}\perp\mathbf{k}, \qquad |\mathbf{B}| = |\mathbf{E}| \tag{3.14}$$

form an orthogonal system and $\mathbf{B}$ is uniquely determined by $\mathbf{E}$ (and vice versa).

At first sight, it may seem that we have been to liberal in formulating the solution (3.13): while the physical electromagnetic field is a *real* vector field, the solutions (3.13) are manifestly complex. The simple solution to this conflict is to identify $\mathrm{Re}\,\mathbf{E}$ and $\mathrm{Re}\,\mathbf{B}$ with the physical fields.[1]



3.3.2 Polarization

In the following we will discuss a number of physically different realizations of plane electromagnetic waves. Since $\mathbf{B}$ is uniquely determined by $\mathbf{E}$, we will focus attention on the latter. Let us choose a coordinate system such that $\mathbf{e}_3 \parallel \mathbf{k}$. We may then write

$$\mathbf{E}(\mathbf{x},t) = (E_1\mathbf{e}_1 + E_2\mathbf{e}_2)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t},$$

where $E_i = |E_i|\exp(i\phi_i)$. Depending on the choice of the complex coefficients E_i, a number of physically different wave–types can be distinguished.

Linearly polarized waves

For identical phases $\phi_1 = \phi_2 = \phi$, we obtain

$$\mathrm{Re}\,\mathbf{E} = (|E_1|\mathbf{e}_1 + |E_2|\mathbf{e}_2)\cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi),$$

i.e. a vector field **linearly polarized** in the direction of the vector $|E_1|\mathbf{e}_1 + |E_2|\mathbf{e}_2$.

▷ Info. Linear polarization is a hallmark of many artificial light sources, e.g. **laser light** is usually linearly polarized. Likewise, the radiation emitted by many **antennae** shows approximately linear polarization.

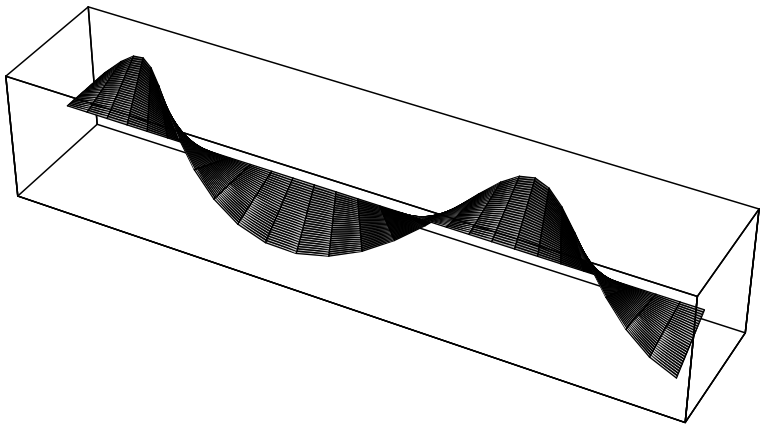


Figure 3.1: Schematic of a circularly polarized electric and magnetic field.

Circularly polarized waves

Next consider the case of a phase difference, $\phi_1 = \phi_2 \mp \pi/2 \equiv \phi$ and equal amplitudes $|E_1| = |E_2| = E$:

$$\mathrm{Re}\,\mathbf{E}(\mathbf{x},t) = E\left(\mathbf{e}_1\cos(\mathbf{k}\cdot\mathbf{x}-\omega t+\phi) \mp \mathbf{e}_2\sin(\mathbf{k}\cdot\mathbf{x}-\omega t+\phi)\right).$$

Evidently, the tip of the vector $\mathbf{E}$ moves on a circle — $\mathbf{E}$ is circularly polarized. Regarded as a function of $\mathbf{x}$ (for any fixed instance of time), $\mathbf{E}$ traces out a spiral whose characteristic period is set by the wave number $\lambda = 2\pi/k$, (see Fig. 3.1.) The sense of orientation of this spiral is determined by the sign of the phase mismatch. For $\phi_1 = \phi_2 + \pi/2$ ($\phi_1 = \phi_2 - \pi/2$) we speak of a right (left) circulary polarized wave or a wave of positive (negative) **helicity**.

▷ Info. As with classical point particles, electromagnetic fields can be subjected to a quantization procedure. In quantum electrodynamics it is shown that the quantae of the electromagnetic field, the **photons** carry definite helicity, i.e. they represent the minimal quantum of circularly polarized waves.

Elliptically polarized waves

Circular and linear polarization represent limiting cases of a more general form of polarization. Indeed, the minimal geometric structure capable of continuously interpolating between a line segment and a circle is the ellipse. To conveniently see the appearance of ellipses, consider the basis change, $\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2)$, and represent the electric field as

$$\mathbf{E}(\mathbf{x},t) = (E_+\mathbf{e}_+ + E_-\mathbf{e}_-)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}.$$

In this representation, a circularly polarized wave corresponds to the limit $E_+ = 0$ (positive helicity) or $E_- = 0$ (negative helicity). Linear polarization is obtained by setting $E_+ = \pm E_-$. It is straightforward to verify that for generic values of the ratio $|E_-/E_+| \equiv r$ one obtains an **elliptically**

[1] One may ask why, then, did we introduce complex notation at all. The reason is that working with exponents of phases is way more convenient than explicitly having to distinguish between the sin and cos functions that arise after real parts have been taken.

polarized wave where the ratio between the major and the minor axis is set by $|1 + r/1 - r|$. The tilting angle α of the ellipse w.r.t. the 1–axis is given by $\alpha = \arg(E_-/E_+)$.

This concludes our discussion of the polarization of electromagnetic radiation. It is important to keep in mind that the different types of polarization discussed above represent limiting cases of what in general is only partially or completely un–polarized radiation. Radiation of a given value of the frequency, ω, usually involves the superposition of waves of different wave vector $\mathbf{k}$ (at fixed wave number $k = |\mathbf{k}| = \omega c^{-1}$.) Only if the amplitudes of all partial waves share a definite phase/amplitude relation, do we obtain a polarized signal. The degree of polarization of a wave can be determined by computing the so–called Stokes parameters. However, we will not discuss this concept in more detail in this text.

3.4 Green function of the wave equation

3.4.1 Computation of the Green function

To prepare our discussion of the full problem (3.10), let us consider the inhomogeneous scalar wave equation

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right)\psi = -4\pi g, \tag{3.15}$$

where the inhomogeneity g and the solution f are scalar function. As with the Poisson equation (2.4) the weak spot of the wave equation is its linearity. We may, therefore, again employ the concept of Green functions to simplify the solution of the problem. The Green function $G(\mathbf{x}, t; \mathbf{x}', t)$ is defined by obvious generalization of the Green function of electrostatics, (2.8) to a problem with space–time degrees of freedom.

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right) G(\mathbf{x}, t; \mathbf{x}', t') = -\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'). \tag{3.16}$$

Once the solution of this equation has been obtained (which requires specification of a set of boundary conditions), the solution of (3.15) becomes a matter of a straightforward integration:

$$f(\mathbf{x}, t) = 4\pi \int d^3x' dt'\, G(\mathbf{x}, t; \mathbf{x}', t') g(\mathbf{x}', t'). \tag{3.17}$$

Assuming vanishing boundary conditions at infinity $G(\mathbf{x}, t; \mathbf{x}', t') \overset{|\mathbf{x}-\mathbf{x}'|,|t-t'|\to\infty}{\longrightarrow} 0$, we next turn to the solution of Eq.(3.16).

▷ Info. In fact, the most elegant and efficient solution strategy utilizes methods of the **theory of complex functions**. Since, however, we do not assume familiarity of the reader with this piece of mathematics, we will use a more elementary technique.

We first note that for the chosen set of boundary conditions, the Green function $G(\mathbf{x} - \mathbf{x}', t - t')$ will depend on the difference of its arguments only. We next Fourier transform Eq.(3.16) in the temporal variable, i.e. we act on the equation with the integral transform $\tilde{f}(\omega) = \frac{1}{2\pi}\int dt\, \exp(i\omega t) f(t)$

(whose inverse is $f(t) = \int dt\, \exp(-i\omega t)\tilde{f}(\omega)$.) The temporally transformed equation is given by

$$\left(\Delta + k^2\right) G(\mathbf{x}, \omega) = -\frac{1}{2\pi}\delta(\mathbf{x}), \tag{3.18}$$

where we defined $k \equiv \omega/c$. If it were not for the constant k (and a trivial scaling factor $(2\pi)^{-1}$ on the l.h.s.) this equation, known in the literature as the **Helmholtz equation**, would be equivalent to the Poisson equation of electrostatics. Indeed, it is straightforward to verify that the solution of (3.18) is given by

$$G^{\pm}(\mathbf{x}, \omega) = \frac{1}{8\pi^2}\frac{e^{\pm ik|\mathbf{x}|}}{|\mathbf{x}|}, \tag{3.19}$$

where the sign ambiguity needs to be fixed on physical grounds.

▷ Math. To **prove Eq.(3.19)**, we introduce polar coordinates centered around $\mathbf{x}'$ and act with the spherical representation of the Laplace operator (cf. section 6.1.2) on $G^{\pm}(r) = \frac{1}{8\pi^2}e^{ikr}/r$. Noting that the radial part of the Laplace operator, $\Delta^{\#}$ is given by $r^{-2}\partial_r r^2 \partial_r = \partial_r^2 + (2/r)\partial_r$ and $\Delta^{\#}(4\pi r)^{-1} = -\delta(\mathbf{x})$ (the equation of the Green function of electrostatics), we obtain

$$\begin{aligned}
\left(\Delta + k^2\right) G^{\pm}(r, \omega) &= \frac{1}{8\pi^2}\left(\partial_r^2 + 2r^{-1}\partial_r + k^2\right)\frac{e^{\pm ikr}}{r} = \\
&= \frac{e^{\pm ikr}}{8\pi^2}\left(\left(\partial_r^2 + 2r^{-1}\partial_r\right) r^{-1} + 2e^{\mp ikr}(\partial_r r^{-1})\partial_r e^{\pm ikr} + e^{\mp ikr} r^{-1}\left(\partial_r^2 + k^2 + 2r^{-1}\partial_r\right) e^{\pm ikr}\right) = \\
&= -\frac{1}{2\pi}\delta(\mathbf{x}) \mp 2ikr^{-2} \pm 2ikr^{-2} = -\frac{1}{2\pi}\delta(\mathbf{x}),
\end{aligned}$$

as required.

Doing the inverse temporal Fourier transform, we obtain

$$G^{\pm}(\mathbf{x}, t) = \int d\omega\, e^{-i\omega t} G_{\omega/c}(\mathbf{x}) = \int d\omega\, e^{-i\omega t}\frac{1}{8\pi^2}\frac{e^{\pm i\omega c^{-1}|\mathbf{x}|}}{|\mathbf{x}|} = \frac{1}{4\pi}\frac{\delta(t \mp c^{-1}|\mathbf{x}|)}{|\mathbf{x}|},$$

or

$$\boxed{G^{\pm}(\mathbf{x} - \mathbf{x}', t - t') = \frac{1}{4\pi}\frac{\delta(t - t' \mp c^{-1}|\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|}}. \tag{3.20}$$

For reasons to become clear momentarily, we call G^+ (G^-) the **retarded (advanced) Green function** of the wave equation.

3.4.2 Physical meaning of the Green function

Retarded Green function

To understand the physical significance of the retarded Green function G^+, we substitute the r.h.s. of Eq.(3.20) into (3.17) and obtain

$$f(\mathbf{x}, t) = \int d^3x' \frac{g(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|}. \tag{3.21}$$

For any fixed instance of space and time, $(\mathbf{x}, t)$, the solution $f(\mathbf{x}, t)$ is affected by the sources $g(\mathbf{x}', t')$ at all points in space and fixed *earlier* times $t' = t - |\mathbf{x} - \mathbf{x}|'/c$. Put differently, a time $t - t' = |\mathbf{x} - \mathbf{x}'|/c$ has to pass before the amplitude of the source $g(\mathbf{x}', t')$ may cause an effect at the observation point $\mathbf{x}$ at time t — the signal received at $(\mathbf{x}, t)$ is subject to a **retardation mechanism**. When the signal is received, it is so at a strength $g(\mathbf{x}', t)/|\mathbf{x} - \mathbf{x}'|$, similarly to the Coulomb potential in electrostatics. (Indeed, for a time independent source $g(\mathbf{x}', t') = g(\mathbf{x}')$, we may enter the wave equation with a time–independent ansatz $f(\mathbf{x})$, whereupon it reduces to the Poisson equation.) Summarizing, the sources act as 'instantaneous Coulomb charges' which are (a) of strength $g(\mathbf{x}', t')$ and felt at times $t = t' + |\mathbf{x} - \mathbf{x}'|/c$.

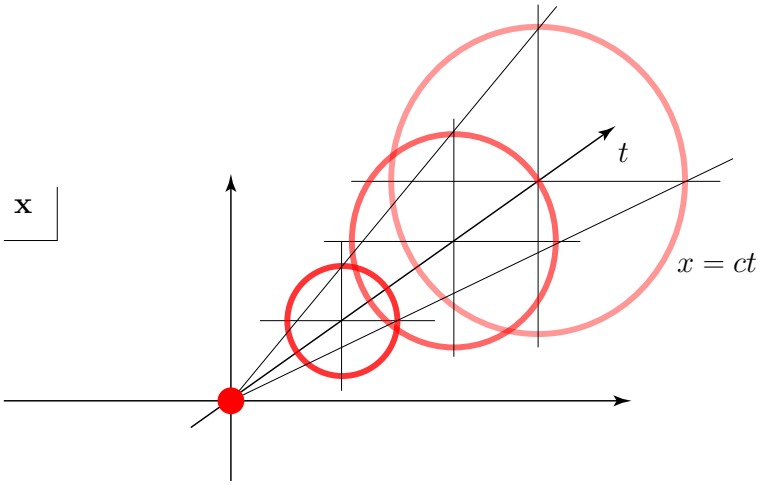


Figure 3.2: Wave front propagation of the pulse created by a point source at the origin schematically plotted as a function of two–dimensional space and time. The width of the front is set by $\delta x = c\delta t$, where δt is the duration of the time pulse. Its intensity decays as $\sim x^{-1}$.

▷ Example: By way of example, consider a point source at the origin, which 'broadcasts' a signal for a short instance of time at $t' \simeq 0$, $g(\mathbf{x}', t') = \delta(\mathbf{x}')F(t')$ where the function F is sharply peaked around $t' = 0$ and describes the temporal profile of the source. The signal is then given by $f(\mathbf{x}, t) = F(t - |\mathbf{x}|/c)/|\mathbf{x}|$, i.e. we obtain a pattern of out–moving **spherical waves**, whose amplitude diminishes as $|\mathbf{x}|^{-1}$ or, equivalently $\sim 1/tc$ (see Fig. 3.2.)

Advanced Green function

Consider now the solution we would have obtained from the advanced Green function, $f(\mathbf{x}, t) = \int d^3x' \frac{g(\mathbf{x}', t+|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}$. Here, the signal responds to the behaviour of the source in *the future*. The principle of cause–and–effect or **causality** is violated implying that the advanced Green function, though mathematically a legitimate solution of the wave equation, does not carry physical meaning. Two more remarks are in order: (a) when solving the wave equation by techniques borrowed from the theory of complex functions, the causality principle is built in from the outset, and the retarded Green function automatically selected. (b) Although the advanced Green function does not carry immanent physical meaning, its not a senseless object altogether. However, the utility of this object discloses itself only in quantum theories.

3.4.3 Electromagnetic gauge field and Green functions

So far we have solved the wave equation for an arbitrary scalar source. Let us now specialize to the wave equations (3.10) for the components of the electromagnetic potential, A_μ. To a first approximation, these are four independent scalar wave equations for the four sources j_μ. We may, therefore, just copy the prototypical solution to obtain the **retarded potentials of electrodynamics**

$$\begin{aligned}\phi(\mathbf{x},t) &= \int d^3x' \, \frac{\rho(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}, \\ \mathbf{A}(\mathbf{x},t) &= \frac{1}{c}\int d^3x' \, \frac{\mathbf{j}(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned} \tag{3.22}$$

There is, however, one important consistency check that needs to be performed: recalling that the wave equations (3.10) hold only in the Lorentz gauge $\partial_\mu A^\mu = 0 \Leftrightarrow c^{-1}\partial_t\phi + \nabla\cdot\mathbf{A} = 0$, we need to check that this condition is actually fulfilled by the solutions (3.22). Relatedly, we have to keep in mind that the sources are not quite independent; they obey the continuity equation $\partial_\mu j^\mu = 0 \Leftrightarrow \partial_t\rho + \nabla\cdot\mathbf{j} = 0$.

As in section 2.3, it will be most convenient to probe the gauge behaviour of the vector potential in a fully developed Fourier language. Also, we will use a four vector notation throughout. In this notation, the Fourier transformation (3.11) assumes the form

$$\tilde{f}(k) = \frac{1}{(2\pi)^4}\int d^4x \, f(x) e^{ix_\mu k^\mu}, \tag{3.23}$$

where a factor of c^{-1} has been absorbed in the integration measure and $k_\mu = (k_0, \mathbf{k})$ with $k_0 = \omega/c$.[2] The Fourier transform of the scalar wave equation (3.15) becomes $k_\mu k^\mu \psi(k) = -4\pi g(k)$. Specifically, the Green function obeys the equation $k_\mu k^\mu G(k) = -(2\pi)^{-4}$ which is solved by $G(k) = -((2\pi)^4 k_\mu k^\mu)^{-1}$. (One may check explicitly that this is the Fourier transform of our solution (3.20), however for our present purposes there is no need to do so.) The solution of the general scalar wave equation (3.17) obtains by *convoluting* the Green function and the source,

$$f(x) = 4\pi(G * g)(x) \equiv 4\pi \int d^4x \, G(x-x')g(x').$$

Using the convolution theorem, this transforms to $f(k) = (2\pi)^4 4\pi \, G(k)g(k) = -\frac{4\pi g(k)}{k_\mu k^\mu}$. Specifically, for the vector potential, we obtain

$$A_\mu = -4\pi \frac{j_\mu}{k_\nu k^\nu}. \tag{3.24}$$

This is all we need to check that the gauge condition is fulfilled: The Fourier transform of the Lorentz gauge relation $\partial_\mu A^\mu = 0$ is given by $k_\mu A^\mu = k^\mu A_\mu = 0$. Probing this relation on (3.24), we obtain $k^\mu A_\mu \propto k^\mu j_\mu = 0$, where we noted that $k^\mu j_\mu = 0$ is the continuity relation.

We have thus shown that the solution (3.22) thus conforms with the gauge constraints. As an important byproduct, our proof above reveals an intimate connection between the gauge behaviour of the electromagnetic potential and current conservation.

Eqs. (3.22) generalize Eqs.(2.12) and (2.53) to the case of dynamically varying sources. In the next sections, we will explore various aspects of the physical contents of these equations.

[2] Recall that $x_0 = ct$, i.e. $k_\mu x^\mu = k_0 x_0 - \mathbf{k}\cdot\mathbf{x} = \omega t - \mathbf{k}\cdot\mathbf{x}$.

3.5 Field energy and momentum

3.5.1 Field energy

In sections 2.2.2 and 3.1, we discussed the energy of static electric and magnetic fields. In this section we will identify the results obtained for the electric and the magnetic field energy, Eqs.(2.20) and (3.2), respectively, as building blocks of a larger picture.

We begin by performing a few basic operations on the Maxwell equations: multiplying Eq.(1.2) by $\mathbf{E}\cdot$ and Eq. (1.3) by $-\mathbf{H}\cdot$, and adding the results to each other, we obtain

$$\mathbf{E}\cdot(\nabla\times\mathbf{H})-\mathbf{H}\cdot(\nabla\times\mathbf{E})-\frac{1}{c}(\mathbf{E}\cdot\partial_t\mathbf{D}+\mathbf{H}\cdot\partial_t\mathbf{B})=\frac{4\pi}{c}\mathbf{j}\cdot\mathbf{E}.$$

Using that for general vector field $\mathbf{v}$ and $\mathbf{w}$, (check!) $\nabla\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{w}\cdot(\nabla\times\mathbf{v})-\mathbf{v}\cdot(\nabla\times\mathbf{w})$, as well as $\mathbf{E}\cdot\partial_t\mathbf{D}=\partial_t(\mathbf{E}\cdot\mathbf{D})/2$ and $\mathbf{H}\cdot\partial_t\mathbf{B}=\partial_t(\mathbf{H}\cdot\mathbf{B})/2$, this equation can be rewritten as

$$\frac{1}{8\pi}\partial_t(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})+\frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{j}\cdot\mathbf{E}=0. \tag{3.25}$$

To understand the significance of Eq.(3.29), let us integrate it over a test volume V:

$$d_t\int_V d^3x\,\frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})=-\int_{S(V)}d\sigma\,\mathbf{n}\cdot\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})-\int_V d^3x\,\mathbf{j}\cdot\mathbf{E}, \tag{3.26}$$

where use of Gauß's theorem has been made. The first term is the sum over the electric and the magnetic field energy density, as derived earlier for static field configurations. We now interpret this term as the energy stored in general electromagnetic field configurations and

$$\boxed{w\equiv\frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})} \tag{3.27}$$

as the **electromagnetic energy density**. What is new in electro*dynamics* is that the field energy may change in time. The r.h.s. of the equation tells us that there are two mechanisms whereby the eletromagnetic field energy may be altered. First, there is a surface integral over the so–called **Poynting vector field**

$$\boxed{\mathbf{S}\equiv\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}.} \tag{3.28}$$

We interpret the integral over this vector as the energy current passing through the surface $\mathbf{S}$ and the Poynting vector field as the **energy current density**. Thus, the pair (energy density, w)/(Poynting vector $\mathbf{S}$) plays a role similar to the pair (charge density, ρ)/(current density, $\mathbf{j}$) for the matter field. However, unlike with matter, the energy of the electromagnetic field is not conserved. Rather, the **balance equation of electromagnetic field energy** above states that

$$\partial_t w+\nabla\cdot\mathbf{S}=-\mathbf{j}\cdot\mathbf{E}. \tag{3.29}$$

The r.h.s. of this equation contains the matter field $\mathbf{j}$ which suggests that it describes the conversion of electromagnetic into mechanical energy. Indeed, the temporal change of the energy of a charged point particle in an electromagnetic field (cf. section 3.1 for a similar line of reasoning) is given by

$d_t U(\mathbf{x}(t)) = -\mathbf{F}\cdot\dot{\mathbf{x}} = -q(\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B})\cdot\mathbf{v} = -q\mathbf{E}\cdot\mathbf{v}$, where we observed that the magnetic component of the Lorentz force is perpendicular to the velocity and, therefore, does not do work on the particle. Recalling that the current density of a point particle is given by $\mathbf{j} = q\delta(\mathbf{x}-\mathbf{x}(t))\mathbf{v}$, this expression may be rewritten as $d_t U = \int d^3x\, \mathbf{j}\cdot\mathbf{E}$. The r.h.s. of Eq. (3.28) is the generalization of this expression to arbitrary current densities.

Energy conservation implies that the work done on a current of charged particles has to be taken from the electromagnetic field. This explains the appearance of the mechanical energy on the r.h.s. of the balance equation (3.29).

3.5.2 Field momentum

Unllike in previous sections on conservation laws, we here identify $\mathbf{B}=\mathbf{H}$, $\mathbf{D}=\mathbf{E}$, *i.e. we consider the vacuum case.*[3]

As a second example of a physical quantity that can be exchanged between matter and electromagnetic fields, we consider momentum. According to Newtons equations, the change of the total mechanical momentum $\mathbf{P}_{\rm mech}$ carried by the particles inside a volume B is given by the integrated force density, i.e.

$$d_t\mathbf{P}_{\rm mech} = \int_B d^3x\,\mathbf{f} = \int_B d^3x\,\left(\rho\mathbf{E}+\frac{1}{c}\mathbf{j}\times\mathbf{B}\right),$$

where in the second equality we inserted the Lorentz force density. Using Eqs. (1.1) and (1.2) to eliminate the sources we obtain

$$d_t\mathbf{P}_{\rm mech} = \frac{1}{4\pi}\int_B d^3x\,\left((\nabla\cdot\mathbf{E})\mathbf{E} - \mathbf{B}\times(\nabla\times\mathbf{B}) + c^{-1}\mathbf{B}\times\dot{\mathbf{E}}\right).$$

Now, using that $\mathbf{B}\times\dot{\mathbf{E}} = d_t(\mathbf{B}\times\mathbf{E}) - \dot{\mathbf{B}}\times\mathbf{E} = d_t(\mathbf{B}\times\mathbf{E}) - c\mathbf{E}\times(\nabla\times\mathbf{E})$ and adding $0 = \mathbf{B}(\nabla\cdot\mathbf{B})$ to the r.h.s., we obtain the symmetric expression

$$d_t\mathbf{P}_{\rm mech} = \frac{1}{4\pi}\int_B d^3x\,\left((\nabla\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E}) + \mathbf{B}(\nabla\cdot\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{B}) + c^{-1}d_t(\mathbf{B}\times\mathbf{E})\right),$$

which may be reorganized as

$$d_t\left(\mathbf{P}_{\rm mech} - \frac{1}{4\pi c}\int_B d^3x\,\mathbf{B}\times\mathbf{E}\right) = \\ = \frac{1}{4\pi}\int_B d^3x\,\left((\nabla\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E}) + \mathbf{B}(\nabla\cdot\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{B})\right).$$

This equation is of the form d_t(something) = (something else). Comparing to our earlier discussion of conservation laws, we are led to interpret the 'something' on the l.h.s. as a conserved quantity. Presently, this quantity is the sum of the total mechanical momentum density $\mathbf{P}_{\rm mech}$ and the integral

$$\boxed{\mathbf{P}_{\rm field} = \int d^3x\,\mathbf{g}, \qquad \mathbf{g}\equiv\frac{1}{4\pi c}\mathbf{E}\times\mathbf{B} = \frac{\mathbf{S}}{c^2}.} \tag{3.30}$$

[3]The discussion of the field momentum in matter (cf. **??**) turns out to be a delicate matter, wherefore we prefer to stay on the safe ground of the vaccum theory.

The structure of this expression suggests to interpret $\mathbf{P}_{\text{field}}$ as the **momentum carried by the electromagnetic field** and $\mathbf{g}$ as the momentum density (which happens to be given by $c^{-2}\times$ the Poynting vector.) If our tentative interpretation of the equation above as a conservation law is to make sense, we must be able to identify its r.h.s. as a surface integral. This in turn requires that the components of the (vector valued) integrand be representable as a $X_j \equiv \partial_i T_{ij}$ i.e. as the divergence of a vector–field $\mathbf{T}_j$ with components T_{ij}. (Here, $j = 1, 2, 3$ plays the role of a spectator index.) If this is the case, we may, indeed, transform the integral to a surface integral, $\int_B d^3x\, X_j = \int_B d^3x\, \partial_i T_{ij} = \int_{\partial B} d\sigma\, \mathbf{n}\cdot\mathbf{T}_j$. Indeed, it is not difficult to verify that

$$[(\nabla\cdot\mathbf{X}) - \mathbf{X}\times(\nabla\times\mathbf{X})]_j = \partial_i\left[X_iX_j - \frac{\delta_{ij}}{2}\mathbf{X}\cdot\mathbf{X}\right]$$

Identifiyng $\mathbf{X} = \mathbf{E}, \mathbf{B}$ and introducing the components of the **Maxwell stress tensor** as

$$\boxed{T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{\delta_{ij}}{2}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B})\right],} \tag{3.31}$$

The r.h.s. of the conservation law assumes the form $\int_B d^3x\, \partial_i T_{ij} = \int_{\partial B} d\sigma\, n_i T_{ij}$, where n_i are the components of the normal vector field of the system boundary. The **law of the conservation of momentum** thus assumes the final form

$$\boxed{d_t(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}})_j = \int_{\partial B} d\sigma\, n_iT_{ij}.} \tag{3.32}$$

Physically, $dtd\sigma n_iT_{ij}$ is the (jth component of the) momentum that gets pushed through $d\sigma$ in time dt. Thus, $d\sigma n_iT_{ij}$ is the momentum per time, or force excerted on $d\sigma$ and n_iT_{ij} the force per area or **radiation pressure** due to the change of linear momentum in the system.

It is straightforward to generalize the discussion above to the conservation of **angular momentum**: The angular momentum carried by a mechanical system of charged particles may be converted into angular momentum of the electromagnetic field. It is evident from Eq. (3.30) that the angular momentum density of the field is given by

$$\mathbf{l} = \mathbf{x}\times\mathbf{g} = \frac{1}{4\pi c}\mathbf{x}\times(\mathbf{E}\times\mathbf{B}).$$

However, in this course we will not discuss angular momentum conservation any further.

3.5.3 Energy and momentum of plane electromagnetic waves

Consider a plane wave in vacuum. Assuming that the wave propagates in 3–direction, the physical electric field, $\mathbf{E}_{\text{phys}}$ is given by

$$\mathbf{E}_{\text{phys.}}(\mathbf{x},t) = \text{Re}\,\mathbf{E}(\mathbf{x},t) = \text{Re}\,\left((E_1\mathbf{e}_1 + E_2\mathbf{e}_2)e^{ikx_3-i\omega t}\right) = r_1u_1(x_3,t)\mathbf{e}_1 + r_2u_2(x_3,t)\mathbf{e}_2,$$

where $u_i(x_3,t) = \cos(\phi_i + kx_3 - \omega t)$ and we defined $E_i = r_i\exp(i\phi_1)$ with real r_i. Similarly, the magnetic field $B_{\text{phys.}}$ is given by

$$\mathbf{B}_{\text{phys.}}(\mathbf{x},t) = \text{Re}\,(\mathbf{e}_3\times\mathbf{E}(\mathbf{x},t)) = r_1u_1(x_3,t)\mathbf{e}_2 - r_2u_2(x,t)\mathbf{e}_1.$$

From these relations we obtain the energy density and Poynting vector as

$$w(\mathbf{x},t) = \frac{1}{8\pi}(\mathbf{E}^2+\mathbf{B}^2) = \frac{1}{4\pi}((r_1u_1)^2+(r_2u_2)^2)(x_3,t)$$
$$\mathbf{S}(\mathbf{x},t) = cw(\mathbf{x},t)\mathbf{e}_3,$$

where we omitted the subscript 'phys.' for notational simplicity.

▷ EXERCISE. Check that w and $\mathbf{S}$ above comply with the conservation law (3.29).

3.6 Electromagnetic radiation

To better understand the physics of the expressions (3.22), let us consider where the sources are confined to within a region in space of typical extension d. Without loss of generality, we assume the time dependence of the sources to be harmonic, with a certain characteristic frequency ω, $j_\mu(x) = j_\mu(\mathbf{x})\exp(-i\omega t)$. (The signal generated by sources of more general time dependence can always be obtained by superposition of harmonic signals.) As we shall see, all other quantities of relevance to us (potentials, fields, etc.) will inherit the same time dependence, i.e. $X(\mathbf{x},t) = X(\mathbf{x})\exp(-i\omega t)$. Specifically, Eq. (3.22) then implies that the electromagnetic potentials, too, oscillate in time, $A_\mu(x) = A_\mu(\mathbf{x})\exp(-i\omega t)$. Substituting this ansatz into Eq. (3.22), we obtain $A_\mu(\mathbf{x}) = \frac{1}{c}\int d^3x' \frac{j_\mu(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$.

As in our earlier analyses of multipole fields, we assume that the observation point $\mathbf{x}$ is far away from the source, $r = |\mathbf{x}| \gg d$. Under these circumstances, we may focus attention on the spatial components of the vector potential,

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int d^3x' \frac{\mathbf{j}(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}, \tag{3.33}$$

where we have substituted the time–oscillatory ansatz for sources and potential into (3.22) and divided out the time dependent phases. From (3.33) the magnetic and electric fields are obtained as

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{E} = ik^{-1}\nabla\times(\nabla\times\mathbf{A}), \tag{3.34}$$

where in the second identity we used the source–free variant of the law of magnetic circulation, $c^{-1}\partial_t\mathbf{E} = -ik\mathbf{E} = \nabla\times\mathbf{B}$.

We may now use the smallness of the ratio d/r to expand $|\mathbf{x}-\mathbf{x}'| \simeq r - \mathbf{n}\cdot\mathbf{x}' + \ldots$, where $\mathbf{n}$ is the unit vector in $\mathbf{x}$–direction. Substituting this expansion into (3.33) and expanding the result in powers of r^{-1}, we obtain a series similar in spirit to the electric and magnetic multipole series discussed above. For simplicity, we here focus attention on the dominant contribution to the series, obtained by approximating $|\mathbf{x}-\mathbf{x}'| \simeq r$. For reasons to become clear momentarily, this term,

$$\mathbf{A}(\mathbf{x}) \simeq \frac{e^{ikr}}{cr}\int d^3x'\,\mathbf{j}(\mathbf{x}'), \tag{3.35}$$

generates **electric dipole radiation**. In section 2.3.2 we have seen that for a static current distribution, the integral $\int\mathbf{j}$ vanishes. However, for a dynamic distribution, we may engage the Fourier

transform of the continuity relation, $i\omega\rho = \nabla \cdot \mathbf{j}$ to obtain

$$\int d^3x\, j_i = \int d^3x\, \nabla x_i \cdot \mathbf{j} = -\int d^3x'\, x_i(\nabla \cdot \mathbf{j}) = -i\omega \int d^3x'\, x_i \rho.$$

Substituting this result into (3.35) and recalling the definition of the electric dipole moment of a charge distribution, $\mathbf{d} \equiv \int d^3x\, \mathbf{x} \cdot \rho$, we conclude that

$$\boxed{\mathbf{A}(\mathbf{x}) \simeq -ik\mathbf{d}\frac{e^{ikr}}{r},} \tag{3.36}$$

is indeed controlled by the electric dipole moment.

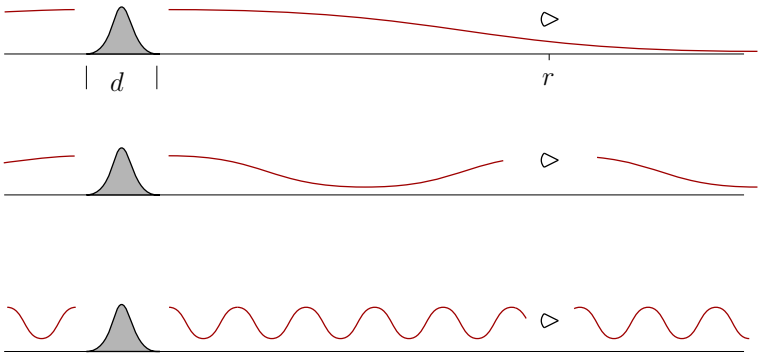


Figure 3.3: Three qualitatively distinct regimes of the radiation field. Discussion, see text.

Besides d and r another characteristic length scale in the problem is the characteristic **wave length** $\lambda \equiv 2\pi/k = 2\pi c/\omega$. For simplicity, we assume that the wave length $\lambda \gg d$ is much larger than the extent of the source.

▷ Info. This latter condition is usually met in practice. E.g. for radiation in the MHz range, $\omega \sim 10^6\,\mathrm{s}^{-1}$ the wave length is given by $\lambda = 3 \cdot 10^8\,\mathrm{m\,s}^{-1}/10^6\,\mathrm{s}^{-1} \simeq 300\,\mathrm{m}$ much larger than the extension of typical antennae.

We then need to distinguish between three different regimes (see Fig. 3.3): the near zone, $d \ll r \ll \lambda$, the intermediate zone, $d \ll \lambda \sim r$, and the far zone, $d \ll \lambda \ll r$. We next discuss these regions in turn.

Near zone

For $r \ll \lambda$ or $kr \ll 1$, the exponential in (3.35) may be approximated by unity and we obtain

$$\mathbf{A}(\mathbf{x}, t) \simeq -i\frac{k}{r}\,\mathbf{d}\, e^{i\omega t},$$

where we have re–introduced the time dependence of the sources. Using Eq.(3.34) to compute the electric and magnetic field, we get

$$\mathbf{B}(\mathbf{x}, t) = i\frac{k}{r^2}\mathbf{n} \times \mathbf{d}e^{i\omega t}, \qquad \mathbf{E} = \frac{\mathbf{n}(\mathbf{n} \cdot \mathbf{d}) - \mathbf{d}}{r^3}e^{i\omega t}.$$

The electric field equals the dipole field (2.16) created by a dipole with time dependent moment $\mathbf{d}\exp(i\omega t)$. This motivates the denotation 'electric dipole radiation'. The magnetic field is by a factor $kr \ll 1$ smaller than the electric field, i.e. in the near zone, the electromagnetic field is dominantly electric. In the limit $k \to 0$ the magnetic field vanishes and the electric field reduces to that of a static dipole.

The analysis of the **intermediate zone**, $kr \sim 1$ is complicated in as much as all powers in an expansion of the exponential in (3.35) in kr must be retained. For a discussion of the resulting field distribution, we refer to[1].

Far zone

For $kr \gg 1$, it suffices to let the derivative operations on the argument kr of the exponential function. Carrying out the derivatives, we obtain

$$\mathbf{B} = k^2 \mathbf{n} \times \mathbf{d} \frac{e^{i(kr-\omega t)}}{r}, \qquad \mathbf{E} = -\mathbf{n} \times \mathbf{B}. \tag{3.37}$$

These asymptotic field distributions have much in common with the vacuum electromagnetic fields above. Indeed, one would expect that far away from the source (i.e. many wavelengths away from the source), the electromagnetic field resembles a spherical wave (by which we mean that the surfaces of constant phase form spheres.) For observation points sufficiently far from the center of the wave, its curvature will be nearly invisible and we the wave will look approximately planar. These expectations are met by the field distributions (3.37): neglecting all derivatives other than those acting on the combination kr (i.e. neglecting corrections of $\mathcal{O}(kr)^{-1}$), the components of $\mathbf{E}$ and $\mathbf{B}$ obey the wave equation (the Maxwell equations in vacuum.[4] The wave fronts are spheres, outwardly propagating at a speed c. The vectors $\mathbf{n} \perp \mathbf{E} \perp \mathbf{B} \perp \mathbf{n}$ form an orthogonal set, as we saw is characteristic for a vacuum plane wave.

Finally, it is instructive to compute the **energy current** carried by the wave. To this end, we recall that the physically realized values of the electromagnetic field obtain by taking the real part of (3.37). We may then compute the Poynting vector (3.28) as

$$\mathbf{S} = \frac{ck^4}{4\pi r^2} \mathbf{n} \left(\mathbf{d}^2 - (\mathbf{n}\cdot\mathbf{d})^2\right) \cos^2(kr - \omega t) \xrightarrow{\langle \ldots \rangle_t} \frac{ck^4}{8\pi r^2} \mathbf{n} \left(\mathbf{d}^2 - (\mathbf{n}\cdot\mathbf{d})^2\right),$$

Where the last expression is the energy current temporally averaged over several oscillation periods ω^{-1}. The energy current is maximal in the plane perpendicular to the dipole moment of the source and decays according to an inverse square law. It is also instructive to compute the **total power** radiated by the source, i.e. the energy current integrated over spheres of constant radius r. (Recall that the integrated energy current accounts for the change of energy inside the reference volume per time, i.e. the power radiated by the source.) Choosing the z–axis of the coordinate system to be colinear with the dipole moment, we have $\mathbf{d}^2 - (\mathbf{n}\cdot\mathbf{d})^2 = d^2 \sin^2\theta$ and

$$P \equiv \int_S d\sigma \mathbf{n} \cdot \mathbf{S} = \frac{ck^4 d^2}{8\pi r^2} r^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \, \sin^2\theta = \frac{ck^4 d^2}{3}.$$

[4] Indeed, one may verify (do it!) that the characteristic factors $r^{-1}e^{i(kr-\omega t)}$ are exact solutions of the wave equations; they describe the space–time profile of a **spherical wave**.

Notice that the radiated power does not depend on the radius of the reference sphere, i.e. the work done by the source is entirely converted into radiation and not, say, in a steadily increasing density of vacuum field energy.

▷ Info. As a concrete example of a radiation source, let us consider a center–fed **linear antenna**, i.a. a piece of wire of length a carrying an AC current that is maximal at the center of the wire. We model the current flow by the prototypical distribution $I(z) = I_0(1 - 2|z|a^{-1})\exp(-i\omega t)$, where we $a \ll \lambda = c/\omega$ and alignment with the z–axis has been assumed. Using the continuity equation, $\partial z I(z,t) + \partial_t \rho(z,t)$, we obtain the charge density (charge per unit length) in the wire as $\rho(z) = iI_0 2(\omega a)^{-1}\,\mathrm{sgn}\,(z)\exp(-i\omega t)$. The dipole moment of the source is thus given by $\mathbf{d} = \mathbf{e}_z \int_{-a}^{a} dz\, z\rho(z,t) = iI_0(a/2\omega)$, and the radiated power by

$$P = \frac{(ka)^2}{12c} I_0^2.$$

The coefficient $R_{\mathrm{rad}} \equiv (ka)^2/12c$ of the factor I_0^2 is called **radiation resistance** of the antenna. To understand the origin of this denotation, notice that $[R_{\mathrm{rad}}] = [c^{-1}]$ has indeed the dimension of resistivity (exercise.) Next recall that the power required to drive a current through a conventional resistor R is given by $P = UI = RI^2$. Comparison with the expression above suggests to interpret R_{rad} as the 'resistance' of the antenna. However, this resistance has nothing to do with dissipative energy losses *inside* the antenna. (I.e. those losses that hold responsible for the DC resistivity of metals.) Rather, work has to be done to feed energy into electromagnetic radiation. This work determines the radiation resistance. Also notice that $R_{\mathrm{rad}} \sim k^2 \sim \omega^2$, i.e. the radiation losses increase quadratic in frequency. This latter fact is of immanent technological importance.

Our results above relied on a first order expansion in the ratio d/r between the extent of the sources and the distance of the observation point. We saw that at this order of the expansion, the source coupled to the electromagnetic field by its electric dipole moment. A more sophisticated description of the field field may be obtained by driving the expansion in d/r to higher order. E.g. at next order, the electric quadrupole moment and the magnetic dipole moment enter the stage. This leads to the generation of **electric quadrupole radiation** and **magnetic dipole radiation**. For an in–depth discussion of these types of radiation we refer to [**?**].

Chapter 4

Macroscopic electrodynamics

4.1 Macroscopic Maxwell equations

Suppose we want to understand the electromagnetic fields in an environment where extended pieces of matter are present. From a purist point of view, we would need to regard the $\mathcal{O}(10^{23})$ carriers of electric and magnetic moments — electrons, protons, neutrons — comprising the medium as sources. Throughout, we will denote the fields $\mathbf{e}, \ldots, \mathbf{h}$ created by this system of sources as **microscopic fields**. (Small characters are used to distinguish the microscopic fields from the effective macroscopic fields to be introduced momentarily.) The 'microscopic Maxwell equations' read as

$$\begin{aligned} \nabla \cdot \mathbf{e} &= 4\pi\rho, \\ \nabla \times \mathbf{b} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{e} &= \frac{4\pi}{c}\mathbf{j}, \\ \nabla \times \mathbf{e} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{b} &= 0, \\ \nabla \cdot \mathbf{b} &= 0. \end{aligned}$$

Now, for several reasons, it does not make much sense to consider these equations as such: First, it is clear that any attempts to get a system of $\mathcal{O}(10^{23})$ highly dynamical sources under control are bound to fail. Second, as a matter of principle, the dynamics of the microscopic sources is governed by quantum effects, i.e. we must not describe them in terms of a classical vector field j. Finally, we aren't even interested in knowing the microscopic fields. Rather, (in classical electrodynamics) we want to understand the behaviour of fields on 'classical' length scales which generally exceed the atomic scales by far.[1] Let us, then, introduce **macroscopic fields** by averaging the microscopic fields as

$$\mathbf{E} \equiv \langle \mathbf{e} \rangle, \qquad \mathbf{B} \equiv \langle \mathbf{b} \rangle,$$

where the averaging procedure is defined by $\langle f(\mathbf{x}) \rangle \equiv \int d^3x' \, f(\mathbf{x} - \mathbf{x}') g(\mathbf{x}')$, and g is a weight function that is unit normalized, $\int gt = 1$ and decays over sufficiently large regions in space. Since the averaging procedure commutes with taking derivatives w.r.t. both space and time, $\partial_{t,\mathbf{x}} \mathbf{E} =$

[1] For example, the wave–length of visible light is about $600\,\mathrm{nm}$, whilst the typical extension of a molecule is $0.1\,\mathrm{nm}$. Thus, 'classical' length scales of interest are about three to four orders of magnitude larger than the microscopic scales.

$\langle \partial_{t,\mathbf{x}} \mathbf{e} \rangle$ (and the same for $\mathbf{B}$), the averaged Maxwell equations read as

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi \langle \rho \rangle, \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} &= \frac{4\pi}{c}\langle \mathbf{j} \rangle, \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

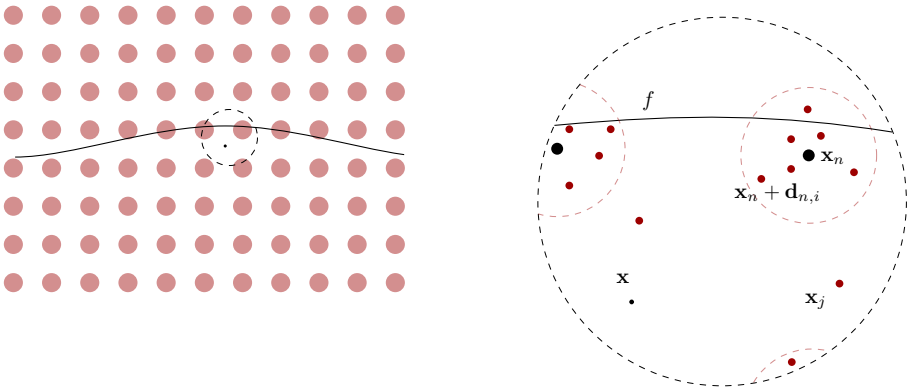


Figure 4.1: Schematic on the spatially averaged system of sources. A typical solid contains spatially localized molecules (indicated by the pattern of spheres) and (optionally) mobile electrons (the tiny inter–atomic dots.) The former can be polarized by external fields. Solid line: Cartoon of the averaging weight function f, centered around a reference point $\mathbf{x}$.

4.1.1 Averaged sources

To better understand the effect of the averaging on the sources, we need to take an (ever so superficial) look into the atomic structure of a typical solid. Generally, two different types of charges in solids have to be distinguished: first, there are charges — nuclei and valence electrons — that are by no more than a vector $\mathbf{a}_{m,j}$ off the center coordinate of a molecule (or atom for that matter) $\mathbf{x}_m$. Second, (in metallic systems) there are free conduction electrons which may abide at arbitrary coordinates $\mathbf{x}_i(t)$ in the system. Denoting the two contributions by ρ_b and ρ_f, respectively, we have $\rho_b(\mathbf{x}, t) = \sum_{m,j} q_{m,j} \delta(\mathbf{x} - \mathbf{x}_m(t) - \mathbf{a}_{m,j}(t))$ and $\rho_f = q_e \sum_i \delta(\mathbf{x} - \mathbf{x}_i(t))$, where $q_{m,j}$ is the charge of the ith molecular constituent and q_e the electron charge.

Averaged charge density and electric susceptibility

The density of the bound charges, averaged over macroscopic proportions, is given by

$$\begin{aligned} \langle \rho_b(\mathbf{x}) \rangle &= \int d\mathbf{x}'\, g(\mathbf{x}') \sum_{m,j} q_{m,j} \delta(\mathbf{x} - \mathbf{x}' - \mathbf{x}_m - \mathbf{a}_{m,j}) = \\ &= \sum_{m,j} g(\mathbf{x} - \mathbf{x}_m - \mathbf{a}_{m,j}) q_{m,j} \simeq \end{aligned}$$

$$\simeq \sum_m q_m g(\mathbf{x}-\mathbf{x}_m) - \nabla \sum_m g(\mathbf{x}-\mathbf{x}_m) \cdot \underbrace{\sum_j \mathbf{a}_{m,j} q_{m,j}}_{\mathbf{d}_m} =$$

$$= \sum_m q_m \langle \delta(\mathbf{x}-\mathbf{x}_m)\rangle - \nabla \cdot \mathbf{P}(\mathbf{x}).$$

In the second line, we used the fact that the range over which the weight function g changes exceeds atomic extensions by far to Taylor expand to first order in the offsets $\mathbf{a}$. The zeroth order term of the expansion is oblivious to the molecular structure, i.e. it contains only the total charge $q_m = \sum_j q_{m,j}$ of the molecules and their center positions. The first order term contains the molecular dipole moments, $\mathbf{d}_m \equiv \sum_j a_{m,j} q_{m,j}$. By the symbol

$$\mathbf{P}(\mathbf{x},t) \equiv \left\langle \sum_m \delta(\mathbf{x}-\mathbf{x}_m(t))\mathbf{d}_m(t) \right\rangle, \tag{4.1}$$

we denote the average **polarization of the medium**. Evidently, $\mathbf{P}$ is a measure of the density of the dipole moments carried by the molecules in the system.

The average density of the mobile carriers in the system is given by $\langle \rho_f(\mathbf{x},t)\rangle = \langle q_e \sum_i \delta(\mathbf{x}-\mathbf{x}_i(t))\rangle$, so that we obtain the **average charge density** as

$$\langle \rho(\mathbf{x},t)\rangle = \rho(\mathbf{x},t) - \nabla \cdot \mathbf{P}(\mathbf{x},t) \tag{4.2}$$

where we defined

$$\rho(\mathbf{x},t) \equiv \left\langle \sum_m q_m \delta(\mathbf{x}-\mathbf{x}_m(t)) + q_e \sum_i \delta(\mathbf{x}-\mathbf{x}_i(t)) \right\rangle$$

to be the effective or macroscopic charge density in the system. Notice that the molecules/atoms present in the medium enter the quantity $\rho(\mathbf{x},t)$ as point–like entities. Their finite polarizability is accounted for by the second term contributing to $\langle\rho\rangle$. Also notice that most solids are electrically neutral on average. I.e. in most cases, ρ will vanish, unless (a) external charges or charged impurities are present in the system, or (b) the charge distribution exhibits large scale collective oscillations.

Substituting Eq. (4.2) into the inhomogeneous Maxwell equations and rearranging terms, we obtain

$$\nabla \cdot (\mathbf{E} + 4\pi\mathbf{P}) = 4\pi\rho.$$

This equation suggests to introduce a field

$$\boxed{\mathbf{D} \equiv \mathbf{E} + 4\pi\mathbf{P}.} \tag{4.3}$$

The sources of this **displacement field** is the net average charge density,

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \tag{4.4}$$

(while the electric field has the spatial average of the full microscopic charge system as its source, $\nabla \cdot \mathbf{E} = 4\pi\langle\rho\rangle$.)

An obvious question to ask is where the medium polarization $\mathbf{P}$ that makes the electric and the displacement field distinct actually comes from. The answer is that in most cases, a finite

polarization is induced by the presence of an electric field, i.e. $\mathbf{P} = \mathbf{P}[\mathbf{E}]$. (Notice the feedback mechanism: an electric leads to polarization which in turn acts as a source for the electric field.)

▷ Info. On a microscopic level, at least two different **mechanisms causing field–induced polarization** need to be distinguished: Firstly, a finite electric field may cause a distortion of the electron clouds surrounding otherwise unpolarized atoms or molecules. The relative shift of the electron clouds against the nuclear centers causes a finite molecular dipole moment which integrates to a macroscopic polarization $\mathbf{P}$. Second, in many substances (mostly of organic chemical origin), the molecules (a) carry a permanent intrinsic dipole moment and (b) are free to move. A finite electric field will cause spatial alignment of the otherwise dis–oriented molecules. This leads to a polarization of the medium as a whole.

Now, external electric fields triggering a non–vanishing polarization are ususally much weaker than the intrinsic microscopic fields keeping the constituents of a solid together. In practice, this means that the polarization may usually be approximated by a *linear* functional of the electric field. The most general form of a linear functional reads as[2]

$$\mathbf{P}(\mathbf{x},t) = \frac{1}{(2\pi)^4}\int d^3x' \int dt'\, \chi(\mathbf{x},t;\mathbf{x}',t')\mathbf{E}(\mathbf{x}',t'), \tag{4.5}$$

where the integral kernel χ is called the **electric susceptibility** of the medium and the factor $(2\pi)^{-4}$ has been introduced for convenience. The susceptibility $\chi(\mathbf{x},t;\mathbf{x}',t')$ describes how an electric field amplitude at $\mathbf{x}'$ and $t' < t$ affects the polarization at $(\mathbf{x},t)$.[3] Notice that the polarization obtains by convolution of χ and $\mathbf{E}$. In Fourier space, the equation assumes the simpler form $\mathbf{P}(q) = \chi(q)\mathbf{E}(q)$, where $q = (\omega/c, \mathbf{q})$ as usual. Accordingly, the electric field and the displacement field are connected by the linear relation

$$\mathbf{D}(q) = \epsilon(q)\mathbf{E}(q), \qquad \epsilon(q) = 1 + 4\pi\chi(q), \tag{4.6}$$

where the function $\epsilon(q)$ is known as the **dielectric function**.

Information on the **frequency/momentum dependence of the dielectric function** has to be inferred from outside electrodynamics, typically from condensed matter physics, or plasma physics.[4] In many cases of physical interest — e.g. in the physics of plasmas or the physics of metallic systems — the dependence of ϵ on both $\mathbf{q}$ and ω has to be taken seriously. Sometimes, only the frequency dependence, or the dependence on the spatial components of momentum is of importance. However, the crudest approximation at all (often adopted in this course) is to approximate the function $\epsilon(q) \simeq \epsilon$ by a constant. (Which amounts to approximating $\chi(\mathbf{x},t;\mathbf{x}',t') \propto \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ by a function infinitely short ranged in space and time.) If not mentioned otherwise, we will thus assume

$$\mathbf{D} = \epsilon\mathbf{E},$$

where ϵ is a material constant.

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[2] *Yet* more generally, one might allow for a non co–linear dependence of the polarization on the field vector:

$$P_i(\mathbf{x},t) = \frac{1}{(2\pi)^4}\int d^3x' \int dt'\, \chi_{ij}(\mathbf{x},t;\mathbf{x}',t')\mathbf{E}_j(\mathbf{x}',t'),$$

where $\chi = \{\chi_{ij}\}$ is a 3×3–matrix field.

[3] By causality, $\chi(\mathbf{x},t;\mathbf{x}',t') \propto \Theta(t-t')$.

[4] A **plasma** is a gas of charged particles.

Averaged current density and magnetic susceptibility

Averaging the current density proceeds in much the same way as the charge density procedure outlined above. As a result of a calculation (see the info block below) that is somewhat more laborious than the one above we obtain

$$\langle \mathbf{j} \rangle \simeq \mathbf{j} + \dot{\mathbf{P}} + c\nabla \times \mathbf{M}, \tag{4.7}$$

where

$$\mathbf{j} \equiv \left\langle q_e \sum \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i) \right\rangle + \left\langle \sum_m q_m \dot{\mathbf{x}}_m \delta(\mathbf{x} - \mathbf{x}_m) \right\rangle$$

is the current carried by the free charge carriers and the point–like approximated molecules, respectively, and

$$\mathbf{M}(\mathbf{x}) = \left\langle \sum_m \delta(\mathbf{x} - \mathbf{x}_m) \frac{1}{2c} \sum_j q_{m,j} \mathbf{a}_{m,j} \times \dot{\mathbf{a}}_{m,j} \right\rangle \tag{4.8}$$

is the average **density of magnetic dipole moments** in the system.

▷ Info. To **compute the average of the microscopic current density**, we we decompose $\mathbf{j} = \mathbf{j}_b + \mathbf{j}_f$ the current density into a bound and a free part. With $\mathbf{j}_b(x) = \sum_{m,j} (\dot{\mathbf{x}}_m + \dot{\mathbf{a}}_{m,j}) \delta(\mathbf{x} - \mathbf{x}_m - \mathbf{a}_{m,j})$, the former is averaged as

$$\begin{aligned}
\langle \mathbf{j}_b(\mathbf{x}) \rangle &= \int d\mathbf{x}'\, g(\mathbf{x}') \sum_{m,j} q_{m,j} (\dot{\mathbf{x}}_m + \dot{\mathbf{a}}_{m,j})\, \delta(\mathbf{x} - \mathbf{x}' - \mathbf{x}_m - \mathbf{a}_{m,j}) = \\
&= \sum_{m,j} g(\mathbf{x} - \mathbf{x}_m - \mathbf{a}_{m,j}) (\dot{\mathbf{x}}_m + \dot{\mathbf{a}}_{m,j}) q_{m,j} \simeq \\
&\simeq \sum_{m,j} q_{m,j} \left[g(\mathbf{x} - \mathbf{x}_m) - \sum_m \nabla g(\mathbf{x} - \mathbf{x}_m) \cdot \mathbf{a}_{m,j} \right] (\dot{\mathbf{x}}_m + \dot{\mathbf{a}}_{m,j}).
\end{aligned}$$

We next consider the different orders in $\mathbf{a}$ contributing to this expansion in turn. At zeroth order, we obtain

$$\langle \mathbf{j}_b(\mathbf{x}) \rangle^{(0)} = \sum_m q_m g(\mathbf{x} - \mathbf{x}_m) \dot{\mathbf{x}}_m = \left\langle \sum_m q_m \dot{\mathbf{x}}_m \delta(\mathbf{x} - \mathbf{x}_m) \right\rangle,$$

i.e. the current carried by the molecules in a point–like approximation. The first order term is given by

$$\langle \mathbf{j}_b(\mathbf{x}) \rangle^{(1)} = \sum_m \left[g(\mathbf{x} - \mathbf{x}_m)\, \dot{\mathbf{d}}_m - (\nabla g(\mathbf{x} - \mathbf{x}_m) \cdot \mathbf{d}_m)\, \dot{\mathbf{x}}_m \right].$$

The form of this expression suggests to compare it with the time derivative of the polarization vector,

$$\dot{\mathbf{P}} = d_t \sum_m g(\mathbf{x} - \mathbf{x}_m) \mathbf{d}_m = \sum_m \left[g(\mathbf{x} - \mathbf{x}_m)\, \dot{\mathbf{d}}_m - (\nabla g(\mathbf{x} - \mathbf{x}_m) \cdot \dot{\mathbf{x}}_m)\, \mathbf{d}_m \right].$$

The difference of these two expressions is given by $\mathbf{X} \equiv \langle \mathbf{j}_b(\mathbf{x}) \rangle^{(1)} - \dot{\mathbf{P}} = \sum_m \nabla g(\mathbf{x} - \mathbf{x}_m) \times (\mathbf{d}_m \times \dot{\mathbf{x}}_m)$. By a dimensional argument, one may show that this quantity is negligibly small: the magnitude $|\dot{\mathbf{x}}_m| \sim v$ is of the order of the typical velocity of the atomic or electronic compounds inside the solid. We thus obtain the rough estimate $|\mathbf{X}(\mathbf{q}, \omega)| \sim v(\nabla \mathbf{P})(\mathbf{q}, \omega)| \sim vq|\mathbf{P}|$, where in the second step we generously

neglected the differences in the vectorial structure of $\mathbf{P}$ and $\mathbf{X}$, resp. However, $|(\dot{\mathbf{P}})(\mathbf{q},\omega)| = \omega|\mathbf{P}(\mathbf{q},\omega)|$. This means that $|\mathbf{X}|/|\dot{\mathbf{P}}| \sim vq/\omega \sim v/c$ is of the order of the ratio of typical velocities of non–relativistic matter and the speed of light. This ratio is so small that it (a) over–compensates the crudeness of our estimates above by far and (b) justifies to neglect the difference $\mathbf{X}$. We thus conclude that

$$\langle \mathbf{j}_b \rangle^{(1)} \simeq \dot{\mathbf{P}}.$$

The second order term contributing to the average current density is given by

$$\langle \mathbf{j}_b \rangle^{(2)} = -\sum_{m,j} q_{m,j}\mathbf{a}_{m,j}\left(\dot{\mathbf{a}}_{m,j}\cdot\nabla g(\mathbf{x}-\mathbf{x}_m)\right)$$

This expression is related to the density of **magnetic dipole moments** carried by the molecules. The latter is given by (cf. Eq. (2.61) and Eq. (4.8))

$$\mathbf{M} = \frac{1}{2c}\sum_{m,j} q_{m,j}\mathbf{a}_{m,j}\times\dot{\mathbf{a}}_{m,j} g(\mathbf{x}-\mathbf{x}_m).$$

As a result of a straightforward calculation, we find that the curl of this expression is given by

$$\nabla\times\mathbf{M} = \frac{1}{c}\langle \mathbf{j}_b \rangle^{(2)} + d_t\frac{1}{2c}\sum_{m,j} q_{m,j}\mathbf{a}_{m,j}\left(\mathbf{a}_{m,j}\cdot\nabla g(\mathbf{x}-\mathbf{x}_m)\right).$$

The second term on the r.h.s. of this expression engages the time derivative of the electric *quadrupole* moments $\sim \sum q a_a a_b$, $a,b = 1,2,3$ carried by the molecules. The fields generated by these terms are arguably very weak so that

$$\langle \mathbf{j}_b \rangle^{(2)} \simeq c\nabla\times\mathbf{M}.$$

Adding to the results above the contribution of the free carriers $\langle \mathbf{j}_f(\mathbf{x})\rangle = q_e\left\langle\sum_i \dot{\mathbf{x}}_i\delta(\mathbf{x}-\mathbf{x}_i)\right\rangle$, we arrive at Eq. (4.7).

Substituting Eq. (4.7) into the second inhomogeneous Maxwell equation, we obtain

$$\nabla\times(\mathbf{B}-4\pi\mathbf{M}) - \frac{1}{c}\partial_t(\mathbf{E}+4\pi\mathbf{P}) = \frac{4\pi}{c}\mathbf{j}.$$

The form of this equation motivates the definition of the **magnetic field**

$$\boxed{\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}.} \tag{4.9}$$

Expressed in terms of this quantity, the Maxwell equation assumes the form

$$\nabla\times\mathbf{H} - \frac{1}{c}\partial_t\mathbf{D} = \frac{4\pi}{c}\mathbf{j}. \tag{4.10}$$

According to this equation, the averaged (or macroscopic) current density acts as a source of the magnetic field. As with the electric sector of the theory, the magnetic fields created by externally imposed macroscopic current distributions (a) 'magnetize' solids, i.e. generate finite fields $\mathbf{M}$, and (b) are generally much weaker than the intrinsic microscopic fields inside a solid. This means that

we can write $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}[\mathbf{H}]$, where $\mathbf{M}[\mathbf{H}]$ may be assumed to be linear in $\mathbf{H}$. In analogy to Eq. (4.5) we thus define

$$\mathbf{M}(\mathbf{x},t) = \frac{1}{(2\pi)^4}\int d^3x' \int dt'\, \chi_m(\mathbf{x},t;\mathbf{x}',t')\mathbf{H}(\mathbf{x}',t'), \tag{4.11}$$

where the function χ is called the **magnetic susceptibility** of the system. The magnetic susceptibility describes how a magnetic field at $(\mathbf{x}',t')$ causes magnetization at $(\mathbf{x},t)$. Everything we said above about the electric susceptibility applies in a similar manner to the magnetic susceptibility. Specifically, we have the Fourier space relation $\mathbf{M}(q) = \chi_m(q)\mathbf{H}(q)$ and

$$\mathbf{B}(q) = \mu(q)\mathbf{H}(q), \qquad \mu(q) = 1 + 4\pi\chi_m(q), \tag{4.12}$$

where $\mu(q)$ is called the **magnetic permeability**. In cases where the momentum dependence of the susceptibility is negligible, we obtain the simplified relation

$$\mathbf{B} = \mu\mathbf{H}, \tag{4.13}$$

where μ is a constant. As in the electric case, a number of different microscopic mechanisms causing deviations of μ off unity can be distinguished. Specifically, for molecules carrying no intrinsic magnetic moment, the presence of an external magnetic field may *induce* molecular currents and, thus, a non–vanishing magnetization. This magnetization is generally directed opposite to the external field, i.e. $\mu < 1$. Substances of this type are called **diamagnetic**. (Even for the most diamagnetic substance known, bismuth, $1-\mu = \mathcal{O}(10^{-4})$, i.e. diamagnetism is a very weak effect.) For molecules carrying a non–vanishing magnetic moment, an external field will cause alignment of the latter and, thus, an effective amplification of the field, $\mu > 1$. Materials of this type are called **paramagnetic**. Typical values of paramagnetic permeabilities are of the order of $\mu - 1 = \mathcal{O}(10^{-5}) - \mathcal{O}(10^{-2})$.

Before leaving this section, it is worthwhile to take a final look at the **general structure of the Maxwell equations in matter**, We note that

- ▷ The averaged charge densities and currents are sources of the electric displacement field $\mathbf{D}$ and the magnetic field $\mathbf{H}$, respectively.
- ▷ These fields are related to the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ by Eqs. (4.6) and (4.12), respectively.
- ▷ The actual fields one would measure in an experiment are $\mathbf{E}$ and $\mathbf{B}$; these fields determine the coupling (fields $\rightsquigarrow$ matter) as described by the Lorentz force law.
- ▷ Similarly, the (matter un–related) homogeneous Maxwell equations contain the fields $\mathbf{E}$ and $\mathbf{B}$.

4.2 Applications of macroscopic electrodynamics

In this sections, we will explore a number of phenomena caused by the joint presence of matter and electromagnetic fields. The organization of the section parallels that of the vacuum part of the course, i.e. we will begin by studying static electric phenomena, then advance to the static magnetic case and finally discuss a few applications in electrodynamics.

4.2.1 Electrostatics in the presence of matter

Generalities

Consider the macroscopic Maxwell equations of electrostatics

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho, \\ \nabla \times \mathbf{E} &= 0, \\ \mathbf{D} &= \epsilon \mathbf{E}, \end{aligned} \tag{4.14}$$

where in the last equation we assumed a dielectric *constant* for simplicity. Now assume that we wish to explore the fields in the vicinity of a boundary separating two regions containing types of matter. In general, the dielectric constants characterizing the two domains will be different, i.e. we have $\mathbf{D} = \epsilon\mathbf{E}$ in region #1 and $\mathbf{D} = \epsilon'\mathbf{E}$ in region #2. (For $\epsilon' = 1$, we describe the limiting case of a matter/vacuum interface.) Optionally, the system boundary may carry a finite surface charge density η.

Arguing as in section 2.2.2, we find that the first and the second of the Maxwell equations imply, respectively,

$$\begin{aligned} (\mathbf{D}(\mathbf{x}) - \mathbf{D}'(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}) &= 4\pi\eta(\mathbf{x}), \\ (\mathbf{E}(\mathbf{x}) - \mathbf{E}'(\mathbf{x})) \times \mathbf{n}(\mathbf{x}) &= 0, \end{aligned} \tag{4.15}$$

where the unprimed/primed quantities refer to the fields in regions #1 and #2, respectively. Thus, the tangential component of the electric field is continuous, while the normal component of the displacement field jumps by an amount set by the charge density.

Example: Dielectric sphere in a homogeneous electric field

To illustrate the computation of electric fields in static environments with matter, consider the example of a massive sphere of radius R and dielectric constant ϵ placed in an homogeneous external displacement field $\mathbf{D}$. We wish to compute the electric field $\mathbf{E}$ in the entire medium.

Since there are no charges present, the electric potential inside and outside the sphere obeys the Laplace equation $\Delta\phi = 0$. Further, choosing polar coordinates such that the z–axis is aligned with the orientation of the external field, we expect the potential to be azimuthally symmetric, $\phi(r, \theta, \phi) = \phi(r, \theta)$. Expressed in terms of this potential, the boundary equations (4.15) assume the form

$$\epsilon\partial_r\phi|_{r=R-0} = \partial_r\phi|_{r=R+0}, \tag{4.16}$$

$$\partial_\theta\phi|_{r=R-0} = \partial_\theta\phi|_{r=R+0}, \tag{4.17}$$

$$\phi|_{r=R-0} = \phi|_{r=R+0}, \tag{4.18}$$

where $R \pm 0 \equiv \lim_{\eta\to 0} R \pm \eta$. To evaluate these conditions, we expand the potential inside and outside the sphere in a series of spherical harmonics, (2.40). Due to the azimuthal symmetry of the problem, only ϕ–independent functions $Y_{l,0} = \sqrt{\frac{2l+1}{4\pi}} P_l$ contribute, i.e.

$$\phi(r, \theta) = \sum_l P_l(\cos\theta) \times \begin{cases} A_l r^l & , r < R, \\ B_l r^l + C_l r^{-(l+1)} & , r \geq R, \end{cases},$$

where we have absorbed the normalization factors $(2l + 1/4\pi)^{1/2}$ in the expansion coefficients A_l, B_l, C_l.

At spatial infinity, the potential must approach the potential $\phi = -Dz = -Dr\cos\theta = -DrP_1(\cos\theta)$ of the uniform external field $\mathbf{D} = D\mathbf{e}_z$. Comparison with the series above shows that $B_1 = -D$ and $B_{l\neq 1} = 0$. To determine the as yet unknown coefficients A_l and C_l, we consider the boundary conditions above. Specifically, Eq. (4.16) translates to the condition

$$\sum_l P_l(\cos\theta)\left(\epsilon l A_l R^{l-1} + (l+1)C_l R^{-(l+2)} + D\delta_{l,1}\right) = 0.$$

Now, the Legendre polynomials are a complete set of functions, i.e. the vanishing of the l.h.s. of the equation implies that all series coefficients must vanish individually:

$$\begin{aligned} &C_0 = 0, \\ &\epsilon A_1 + 2R^{-3}C_1 + D = 0, \\ &\epsilon l A_l R^{l-1} + (l+1)C_l R^{-(l+2)} = 0, \qquad l > 1. \end{aligned}$$

A second set of equations is obtained from Eq. (4.17): $\partial_\theta \sum_l P_l(\cos\theta)((A_l - B_l)R^l - C_l R^{-(l+1)}) = 0$ or $\sum_{l>0} P_l(\cos\theta)((A_l - B_l)R^l - C_l R^{-(l+1)}) = \text{const.}$. The second condition implies (think why!) $(A_l - B_l)R^l - C_l R^{-(l+1)} = 0$ for all $l > 0$. Substituting this condition into Eq. (4.18), we finally obtain $A_0 = 0$. To summarize, the expansion coefficients are determined by the set of equations

$$\begin{aligned} &A_0 = 0, \\ &(A_1 + D)R - C_1 R^{-2} = 0, \\ &A_l R^l - C_l R^{-(l+1)} = 0, \qquad l > 1. \end{aligned}$$

It is straightforward to verify that these equations are equivalent to the conditions

$$C_1 = DR^3\frac{\epsilon - 1}{\epsilon + 2}, \qquad A_1 = -\frac{3}{\epsilon + 2}D,$$

while $C_{l>1} = A_{l>1} = 0$. The potential distorted by the presence of the sphere is thus given by

$$\phi(r,\theta) = -rE_0\cos\theta \times \begin{cases} \dfrac{3}{\epsilon+2} & , r < R \\ 1 - \dfrac{\epsilon - 1}{\epsilon + 2}\left(\dfrac{R}{r}\right)^3 & , r \geq R, \end{cases} \tag{4.19}$$

The result (4.19) affords a very intuitive physical interpretation:

- ▷ Inside the sphere the electric field $\mathbf{E} = -\nabla\phi = \nabla E_0 z\frac{3}{\epsilon+2} = E_0\frac{3}{\epsilon+2}\mathbf{e}_z$ is parallel to the external electric field, but weaker in magnitude (as long as $\epsilon > 1$ which usually is the case.) This indicates the buildup of a polarization $\mathbf{P} = \frac{\epsilon-1}{4\pi}\mathbf{E} = \frac{3}{4\pi}\frac{\epsilon-1}{\epsilon+2}\mathbf{E}_0$.

  Outside the sphere, the electric field is given by a superposition of the external field and the field generated by a dipole potential $\phi = \mathbf{d}\cdot\mathbf{x}/r^3 = d\cos\theta/r^2$, where the dipole moment is given by $\mathbf{d} = V\mathbf{P}$ and $V = 4\pi R^3/3$ is the volume of the sphere.

  To understand the origin of this dipole moment, notice that in a polarizable medium, and in the absence of external charges, $0 = \nabla\cdot\mathbf{D} = \nabla\cdot\mathbf{E} + 4\pi\nabla\cdot\mathbf{P}$, while the microscopic charge

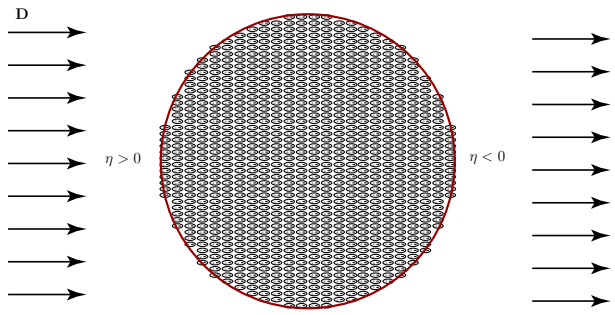


Figure 4.2: Polarization of a sphere by an external electric field.

density $\nabla \cdot \mathbf{E} = 4\pi\rho_{\rm mic}$ need not necessarily vanish. In exceptional cases where $\rho_{\rm mic}$ does not vanish upon averaging, we denote $\langle\rho_{\rm mic}\rangle = \rho_{\rm pol}$ the **polarization charge**. The polarization charge is determined by the condition

$$\nabla \cdot \mathbf{P} = -\rho_{\rm pol}.$$

Turning back to the sphere, the equation $0 = \nabla \cdot \mathbf{D} = \frac{\epsilon}{\chi}\nabla \cdot \mathbf{P}$ tells us that there is no polarization charge in the bulk of the sphere (nor, of course, outside the sphere.) However, arguing as in section 2.2.2, we find that the sphere (or any boundary between two media with different dielectric constants) carries a **surface polarization charge**

$$\eta_{\rm ind} = P_n' - P_n.$$

Specifically, in our problem, $P_n' = 0$ so that $\eta = -P_n = -\frac{3}{4\pi}\frac{\epsilon-1}{\epsilon+2}E_0\cos\theta$. Qualitatively, the origin of this charge is easy enough to understand: while in the bulk of the sphere, the induced dipole moments cancel each other upon spatial averaging (see the figure), uncompensated charges remain at the surface of the medium. These surface charges induce a finite and quite macroscopic dipole moment which is aligned opposite to $\mathbf{E}_0$ and acts as a source of the *electric* field (but not of the displacement field).

4.2.2 Magnetostatics in the presence of matter

Generalities

The derivation of 'magnetic boundary conditions' in the presence of materials with a finite permeability largely parallels the electric case: the macroscopic equations of magnetostatics read as

$$\begin{aligned} \nabla \times \mathbf{H} &= \frac{4\pi}{c}\mathbf{j}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \mathbf{H} &= \mu^{-1}\mathbf{B}. \end{aligned} \tag{4.20}$$

Again, we wish to explore the behaviour of the fields in the vicinity of a boundary between two media with different permeabilities. Proceeding in the usual manner, i.e. by application of infinitesimal

variants of Gauß's and Stokes law, respectively, we derive the equations

$$\begin{aligned}(\mathbf{B}(\mathbf{x}) - \mathbf{B}'(\mathbf{x})) \cdot \mathbf{n} &= 0, \\ (\mathbf{H}(\mathbf{x}) - \mathbf{H}'(\mathbf{x})) \times \mathbf{n} &= \frac{4\pi}{c}\mathbf{j}_s,\end{aligned} \tag{4.21}$$

where $\mathbf{j}_s$ is an optional surface current.

Example: paramagnetic sphere in a homogeneous magnetic field

Consider a sphere of radius R and permeability $\mu > 1$ in the presence of an external magnetic field $\mathbf{H}$. We wish to compute the magnetic induction $\mathbf{B}$ inside and outside the sphere. In principle, this can be done as in the analogous problem on the dielectric sphere, i.e. by Legendre polynomial series expansion. However, comparing the magnetic Maxwell equations to electric Maxwell equations considered earlier,

$$\begin{aligned}\nabla \cdot \mathbf{B} = 0 \qquad &\leftrightarrow \qquad \nabla \cdot \mathbf{D} = 0, \\ \nabla \times \mathbf{H} = 0 \qquad &\leftrightarrow \qquad \nabla \times \mathbf{E} = 0, \\ \mathbf{B} = \mu\mathbf{H} \qquad &\leftrightarrow \qquad \mathbf{D} = \epsilon\mathbf{E}, \\ \mathbf{P} = \frac{\epsilon - 1}{4\pi}\mathbf{E} \qquad &\leftrightarrow \qquad \mathbf{M} = \frac{\mu - 1}{4\pi}\mathbf{H},\end{aligned}$$

we notice that there is actually no need to do so: Formally identifying $\mathbf{B} \leftrightarrow \mathbf{D}, \mathbf{H} \leftrightarrow \mathbf{E}, \mu \leftrightarrow \epsilon, \mathbf{M} \leftrightarrow \mathbf{P}$, the two sets of equation become equivalent, and we may just copy the solution obtained above. Thus (i) the magnetic field outside the sphere is a superposition of the external field and a magnetic dipole field, where (ii) the dipole moment $\mathbf{M} = \frac{3}{4\pi}\frac{\mu-1}{\mu+2}\mathbf{H}$ is parallel to the external field. This magnetic moment is caused by the orientation of the intrinsic moments along the external field axis; consequently, the actually felt magnetic field (the magnetic induction) $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ exceeds the external field.

4.3 Wave propagation in media

What happens if an electromagnetic wave impinges upon a medium characterized by a non–trivial dielectric function ϵ?[5] And how does the propagation behaviour of waves relate to actual physical processes inside the medium? To prepare the discussion of these questions, we will first introduce a physically motivated model for thhe dependence of the dielectric function on its arguments. This modelling will establish a concrete link between the amplitude of the electromagnetic waves and the dynamics of the charge carriers inside the medium.

4.3.1 Model dielectric function

In Fourier space, the dielectric function, $\epsilon(\mathbf{q}, \omega)$ is a function of both wave vector and frequency. It owes its frequency dependence to the fact that a wave in matter may prompt the creation of excitations of the molecular degrees of freedom. The feedback of these excitations to the electromagnetic

[5] We emphasize dielectric function because, as shall be seen below, the magnetic permeability has a much lesser impact on electromagnetic wave propagation.

wave is described by the frequency dependence of the function ϵ. To get an idea of the relevant frequency scales, notice that typical excitation energies in solids are of the order $\hbar\omega < 1\,\mathrm{eV}$. Noting that Planck's constant $\hbar \simeq 0.6 \times 10^{-15}\,\mathrm{eVs}$, we find that the characteristic frequencies at which solids dominantly respond to electromagnetic waves are of the order $\omega < 10^{15}\mathrm{s}^{-1}$. However, the wave lengths corresponding to these frequencies, $\lambda = 2\pi/|\mathbf{q}| \sim 2\pi c/\omega \sim 10^{-6}\,\mathrm{m}$ are much larger than the range of a few interatomic spacings over which we expect the non–local feedback of the external field into the polarization to be 'screened' (the range of the susceptibility χ.) This means (think why!) that the function $\epsilon(\mathbf{q},\omega) \simeq \epsilon(\omega)$ is largely momentum–independent at the momentum scales of physical relevance.

To obtain a crude model for the function $\epsilon(\omega)$, we imagine that the electrons sourrounding the molecules inside the solid are harmonically bound to the nuclear center coordinates. An individual electron at spatial coordinate $\mathbf{x}$ (measured w.r.t. a nuclear position) will then be subject to the equation of motion

$$m\left(d_t^2 + \omega_0^2/2 + \gamma d_t\right)\mathbf{x}(t) = e\mathbf{E}(t), \tag{4.22}$$

where ω_0 is the characteristic frequency of the oscillator motion of the electron, γ a relaxation constant, and the variation of the external field over the extent of the molecule has been neglected (cf. the discussion above.) Fourier transforming this equation, we obtain $\mathbf{d} \equiv e\mathbf{x}(\omega) = e^2\mathbf{E}(\omega)m^{-1}/(\omega_0^2 - \omega^2 - i\gamma\omega)$. The dipole moment $d_m(\omega)$ of an entire molecule with Z electrons f_i of which oscillate with frequency ω_i and damping rate γ_i ($\sum_i f_i = Z$) is given by $\mathbf{d}_m(\omega) = \sum_i f_i \mathbf{d}_{(\omega_0\to\omega_i,\gamma\to\gamma_i)}$. Denoting the density of molecules in the by n, we thus obtain (cf. Eqs. (4.1), (4.3), and (4.6))

$$\epsilon(\omega) = 1 + \frac{4\pi n e^2}{m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega}. \tag{4.23}$$

With suitable quantum mechanical definitions of the material constants f_i, ω_i and γ_i, the **model dielectric function** (4.23) provides an accurate description of the molecular contribution to the dielectric behaviour of solids. A schematic of the typical behaviour of real and imaginary part of the dielectric function is shown in Fig. 4.3. A few remarks on the profile of these functions:

- ▷ The material constant ϵ employed in the 'static' sections of this chapter is given by $\epsilon = \epsilon(0)$. The imaginary part of the zero frequency dielectric function is negligible.
- ▷ For reasons to become clear later on, we call the function

  $$\alpha(\omega) = \frac{2\omega}{c}\,\mathrm{Im}\,\sqrt{\mu\epsilon(\omega)} \tag{4.24}$$

  the (frequency dependent) **absorption coefficient** and

  $$n(\omega) = \mathrm{Re}\,\sqrt{\mu\epsilon(\omega)} \tag{4.25}$$

  the (frequency dependent) **refraction coefficient** of the system.
- ▷ For most frequencies, i.e. everywhere except for the immediate vicinity of a resonant frequency ω_i, the absorption coefficient is negligibly small. In these regions, $d_\omega\,\mathrm{Re}\,\epsilon(\omega) > 0$. The positivity of this derivative defines a region of **normal dispersion**.

▷ In the immediate vicinity of a resonant frequency, $|\omega - \omega_i| < \gamma_i/2$, the absorption coefficient shoots up while the derivative $d_\omega \operatorname{Re}\epsilon(\omega) < 0$ changes sign. As we shall discuss momentarily, the physics of these frequency windows of **anomalous dispersion** is governed by resonant absorption processes.

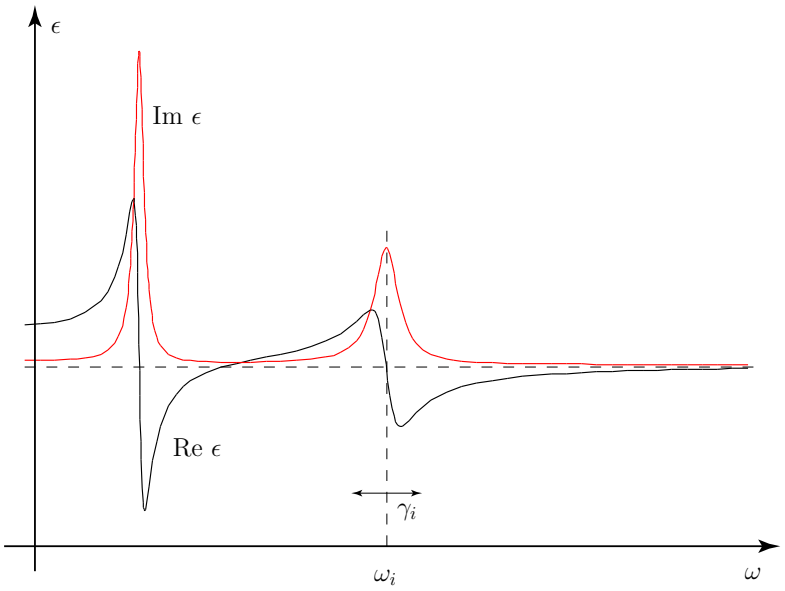


Figure 4.3: Schematic of the functional profile of the function $\epsilon(\omega)$. Each resonance frequency ω_i is center of a peak in the imaginary part of ϵ and of a resonant swing of in the real part. The width of these structures is determined by the damping rate γ_i.

4.3.2 Plane waves in matter

To study the impact of the non–uniform dielectric function on the behaviour of electromagnetic waves in matter, we consider the spatio–temporal Fourier transform of the Maxwell equations in a medium free of extraneous sources, $\rho = 0$, $\mathbf{j} = 0$. Using that $\mathbf{D}(k) = \epsilon(k)\mathbf{E}(k)$ and $\mathbf{H}(k) = \mu^{-1}(k)\mathbf{B}(k)$, where $k = (\omega/c, \mathbf{k})$, we have

$$\begin{aligned} \mathbf{k}\cdot(\epsilon\mathbf{E}) &= 0, \\ \mathbf{k}\times(\mu^{-1}\mathbf{B}) + \frac{\omega}{c}(\epsilon\mathbf{E}) &= 0, \\ \mathbf{k}\times\mathbf{E} - \frac{\omega}{c}\mathbf{B} &= 0, \\ \mathbf{k}\cdot\mathbf{B} &= 0, \end{aligned}$$

where we omitted momentum arguments for notational clarity. We next compute $\mathbf{k}\times$(third equation) $+$ $(\omega/c)\times$(second equation) and $(\omega/c)\times$(third equation) $-$ $\mathbf{k}\times$(second equation) to obtain the **Fourier representation of the wave equation**

$$\left(k^2 - \frac{\omega^2}{c^2(\omega)}\right) \times \begin{Bmatrix} \mathbf{E}(\mathbf{k},\omega) \\ \mathbf{B}(\mathbf{k},\omega) \end{Bmatrix} = 0, \tag{4.26}$$

where

$$\boxed{c(\omega) = \frac{c}{\sqrt{\mu\epsilon(\omega)}}}, \tag{4.27}$$

is the **effective velocity of light in matter**.

Comparing with our discussion of section 3.3, we conclude that a **plane electric wave in a matter** is mathematically described by the function

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 \exp(i(\mathbf{k}\cdot\mathbf{x} - \omega t)),$$

where $\mathbf{k} = k\mathbf{n}$ and $k = \omega/c(\omega)$ is a (generally complex) function of the wave frequency. Specifically, for $\mathrm{Im}\,(\mathbf{k}\cdot\mathbf{x}) > 0$, the wave is will be exponentially damped. Splitting the wave number k into its real and imaginary part, we have

$$k = \mathrm{Re}\,k + i\,\mathrm{Im}\,k = \frac{\omega}{c}n(\omega) + \frac{i}{2}\alpha(\omega),$$

where n and α are the refraction and absorption coefficient, respectively. To better understand the meaning of the absorption coefficient, let us compute the Poynting vector of the electromagnetic wave. As in section 3.3, the Maxwell equation $\nabla\cdot\mathbf{D} = \epsilon\nabla\cdot\mathbf{E} \propto \mathbf{k}\cdot\mathbf{E}_0$ implies transversality of the electric field. (Here we rely on the approximate $\mathbf{x}$–independence of the dielectric function.) Choosing coordinates such that $\mathbf{n} = \mathbf{e}_3$ and assuming planar polarization for simplicity, we set $\mathbf{E}_0 = E_0\mathbf{e}_1$ with a real coefficient E_0. The — physically relevant — real part of the electric field is thus given by

$$\mathrm{Re}\,\mathbf{E} = E_0\mathbf{e}_1\cos(\omega(zn/c - t))e^{-\alpha z/2}.$$

From the Maxwell equations $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{E} + c^{-1}\partial_t\mathbf{B}$, we further obtain (exercise) $\mathrm{Re}\,\mathbf{B} = E_0\mathbf{e}_2\cos(\omega(zn/c - t))e^{-\alpha z/2}$. Assuming that the permeability is close to unity, $\mu \simeq 1$ or $\mathbf{B} \simeq \mathbf{H}$, we thus obtain for the magnitude of the Poynting vector

$$|\mathbf{S}| = \frac{c}{4\pi}|\mathbf{E}\times\mathbf{H}| = \frac{cE_0^2}{4\pi}\cos^2(\omega(zn/c - t))e^{-\alpha z} \xrightarrow{\langle\ldots\rangle_t} \frac{cE_0^2}{8\pi}e^{-\alpha z},$$

where in the last step we have averaged over several intervals of the oscillation period $2\pi/\omega$. According to this result,

> The electromagnetic energy current inside a medium decays at a rate set by the absorption coefficient.

This phenomenon is easy enough to interpret: according to our semi–phenomenological model of the dielectric function above, the absorption coefficient is non–vanishing in the immediate vicinity of a resonant frequency ω_i, i.e. a frequency where molecular degrees of freedom oscillate. The energy stored in an electromagnetic wave of this frequency may thus get converted into mechanical oscillator energy. This goes along with a loss of field intensity, i.e. a diminishing of the Poynting vector or, equivalently, a loss of electromagnetic energy density, w.

4.3.3 Dispersion

In the previous section we have focused on the imaginary part of the dielectric function and on its attenuating impact on individual plane waves propagating in matter. We next turn to the discussion of the — equally important — role played by the real part. To introduce the relevant physical principles in a maximally simple environment, we will focus on a one–dimensional model of wave propagation throughout. Consider, thus, the one–dimensional variant of the wave equations (4.26),

$$\left(k^2 - \frac{\omega^2}{c^2(\omega)}\right)\psi(k,\omega) = 0, \tag{4.28}$$

where k is a one–dimensional wave 'vector' and $c(\omega) = c/\sqrt{\epsilon(\omega)}$ as before. (For simplicity, we set $\mu = 1$ from the outset.) Plane wave solutions to this equation are given by $\psi(x,t) = \psi_0 \exp(ik(\omega)x - i\omega t)$, where $k(\omega) = \omega/c(\omega)$.[6] Notice that an alternative representation of the plane wave configurations reads as $\psi(x,t) = \psi_0 \exp(ikx - i\omega(k)t)$, where $\omega(k)$ is defined implicitly, viz. as a solution of the equation $\omega/c(\omega) = k$.

To understand the consequences of frequency dependent variations in the real part of the dielectric function, we need, however, to go beyond the level of isolated plane wave. Rather, let us consider a superposition of plane waves (cf. Eq. (3.12)),

$$\psi(x,t) = \int dk\, \psi_0(k) e^{-i(kx - \omega(k)t)}, \tag{4.29}$$

where $\psi_0(k)$ is an arbitrary function. (Exercise: check that this superposition solves the wave equations.)

Specifically, let us chose the function $\psi_0(k)$ such that, at initial time $t = 0$, the distribution $\psi(x, t = 0)$ is localized in a finite region in space. Configurations of this type are called **wave packets**. While there is a lot of freedom in choosing such spatially localizing envelope functions, a convenient (and for our purposes sufficiently general) choice of the weight function ψ_0 is a Gaussian,

$$\psi_0(k) = \psi_0 \exp\left(-(k - k_0)^2 \frac{\xi^2}{4}\right),$$

where $\psi_0 \in \mathbb{C}$ is a amplitude coefficient fixed and ξ a coefficient of dimensionality 'length' that determines the spatial extent of the wave package (at time $t = 0$.) To check that latter assertion, let us compute the spatial profile $\psi(x,0)$ of the wave package at the initial time:

$$\begin{aligned}\psi(x,0) = \psi_0 \int dk\, e^{-(k-k_0)^2 \frac{\xi^2}{4} - ikx} = \psi_0 e^{ik_0 x} \int dk\, e^{-k^2 \frac{\xi^2}{4} - ikx} = \\ = \psi_0 e^{ik_0 x} \int dk\, e^{-k^2 \frac{\xi^2}{4} - (x/\xi)^2} = \frac{\sqrt{\pi}}{2} \psi_0 \xi\, e^{ik_0 x} e^{-(x/\xi)^2}.\end{aligned}$$

The function $\psi(x,0)$ is concentrated in a volume of extension ξ; it describes the profile of a small wave 'packet' at initial time $t = 0$. What will happen to this wave packet as time goes on?

[6] There is also the 'left moving' solution, $\psi(x,t) = \psi_0 \exp(ik(\omega)x + i\omega t)$, but for our purposes it will be sufficient to consider right moving waves.

To answer this question, let us assume that the scales over which the function $\omega(k)$ varies are much smaller than the extension of the wave package in wave number space, k. It is, then, a good idea to expand $\omega(k)$ around k_0:

$$\omega(k) = \omega_0 + v_g(k-k_0) + \frac{a}{2}(k-k_0)^2 + \ldots,$$

where we introduced the abbreviations $\omega_0 \equiv \omega(k_0)$, $v_g \equiv \omega'(k_0)$ and $a \equiv \omega''(k_0)$. Substituting this expansion into Eq. (4.29) and doing the Gaussian integral over wave numbers we obtain

$$\begin{aligned}\psi(x,t) &= \psi_0 \int dk\, e^{-(k-k_0)^2\xi^2/4 - i(kx-\omega(k)t)} \simeq \\ &\simeq \psi_0 e^{ik_0(x-v_p t)} \int dk\, e^{-(\xi^2+2iat)(k-k_0)^2/4 - i(k-k_0)(x-v_g t)} = \\ &\simeq \frac{\sqrt{\pi}\psi_0}{2}\xi(t) e^{ik_0(x-v_p t)} e^{-((x-v_g t)/\xi(t))^2},\end{aligned}$$

where we introduced the function $\xi(t) = \sqrt{\xi^2 + 2iat}$, and the abbreviation $v_p \equiv \omega(k_0)/k_0$

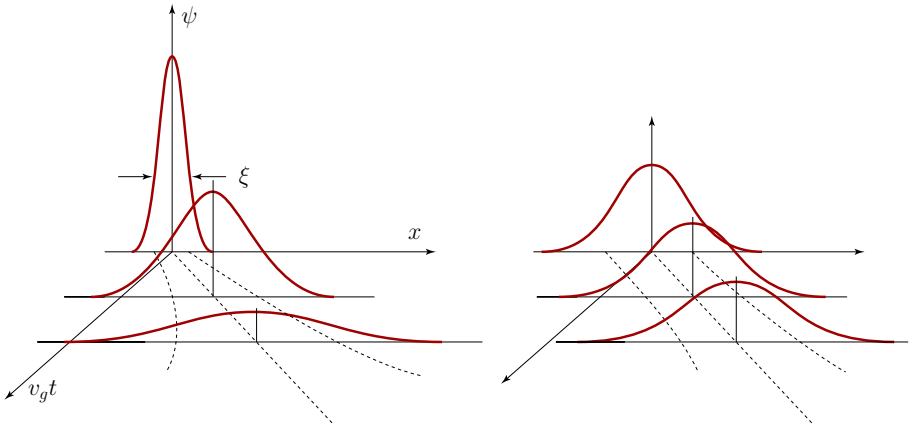


Figure 4.4: Left: Dispersive propagation of an initially sharply focused wave package. Right: A more shallow wave package suffers less drastically from dispersive deformation.

Let us try to understand the main characteristics of this result:

▷ The center of the wave package moves with a characteristic velocity $v_g = \partial_k \omega(k_0)$ to the right. In as much as v_g determines the effective center velocity of a superposition of a continuum or 'group' of plain waves, v_g is called the **group velocity** of the system. Only in vacuum where $c(\omega) = c$ is independent of frequency we have $\omega = ck$ and the group velocity $v_g = \partial_k \omega = c$ coincides with the vacuum velocity of light.

A customary yet far less useful notion is that of a **phase velocity**: an individual plane wave behaves as $\sim \exp(i(kx - \omega(k)t)) = \exp(ik(x - (\omega(k)/k)t))$. In view of the structure of the exponent, one may call $v_p \equiv \omega(k)/k$ the 'velocity' of the wave. Since, however, the wave extends over the entire system anyway, the phase velocity is not of direct physical relevance. Notice that the phase velocity $v_p = \omega(k)/k = c(k) = c/n(\omega)$, where the refraction coefficient has been introduced in (4.25). For most frequencies and in almost all substances, $n > 1 \Rightarrow$

$v_p < c$. In principle, however, the phase velocity may become larger than the speed of light.[7] The group velocity $v_g = \partial_k \omega(k) = (\partial_\omega k(\omega))^{-1} = (\partial_\omega \omega n(\omega))^{-1} c = (n(\omega) + \omega \partial_\omega n(\omega))^{-1} c = v_p (1 + n^{-1} \omega \partial_\omega n)^{-1}$. In regions of normal dispersion, $\partial_\omega n > 0$ implying that $v_g < v_p$. For the discussion of anomalous cases, where v_g may exceed the phase velocity or even the vacuum velocity of light, see [1].

▷ The **width of the wave package**, Δx changes in time. Inspection of the Gaussian factor controlling the envelope of the packet shows that $\mathrm{Re}\,\psi \sim e^{(x-v_g t)^2 (\xi^{-2} + \xi^{*-2})/2}$, where the symbol '$\sim$' indicates that only the exponential dependence of the package is taken into account. Thus, the width of the package is given by $\xi(t) = \xi(1 + (2at/\xi^2)^2)^{1/2}$.

According to this formula, the rate at which $\xi(t)$ increases (the wave package flows apart) is the larger, the sharper the spatial concentration of the initial wave package was (cf. Fig. 4.4). For large times, $t \gg \xi^2/a$, the width of the wave package increases as $\Delta x \sim t$. The disintegration of the wave package is caused by a phenomenon called **dispersion**: The form of the package at $t = 0$ is obtained by (Fourier) superposition of a large number of plane waves. If all these plane waves propagated with the same phase velocity (i.e. if $\omega(k)$ was k–independent), the form of the wave package would remain un–changed. However, due to the fact that in a medium plane waves of different wave number k generally propagate at different velocities, the Fourier spectrum of the package at non–zero times differs from that at time zero. Which in turn means that the integrity of the form of the wave package gets gradually lost.

The dispersive spreading of electromagnetic wave packages is a phenomenon of immense applied importance. For example, dispersive deformation is one of the main factors limiting the information load that can be pushed through **fiber optical cables**. (For too high loads, the 'wave packets' constituting the bit stream through such fibers begin to overlap thus loosing their identity. The construction of ever more sophisticated counter measures optimizing the data capacity of optical fibers represents a major stream of applied research.

4.3.4 Electric conductivity

As a last phenomenon relating to macroscopic electrodynamics, we discuss the electric conduction properties of metallic systems.

Empirically, we know that in a metal, a finite electric field will cause current flow. In its most general form, the relation between field and current assumes a form similar to that between field and polarization discussed above:

$$\boxed{j_i(\mathbf{x}, t) = \int d^3x' \int_{-\infty}^{t} dt'\, \sigma_{ij}(\mathbf{x} - \mathbf{x}', t - t') E_j(\mathbf{x}', t'),} \tag{4.30}$$

where $\sigma = \{\sigma_{ij}(\mathbf{x}, t)\}$ is called the **conductivity tensor**. This equation states that a field may cause current flow at different spatial locations in time. Equally, a field may cause current flow in *directions*

[7] However, there is no need to worry that such anomalies conflict with Einsteins principle of relativity to be discussed in chapter 5 below: the dynamics of a uniform wave train is not linked to the transport of energy or other observable physical quantities, i.e. there is no actual physical entity that is transported at a velocity larger than the speed of light.

different from the field vector. (For example, in a system subject to a magnetic field in z–direction, an electric field in x–direction will cause current flow in both x– and y–direction, why?) As with the electric susceptibility, the non–local space dependence of the conductivity may often be neglected, $\sigma(\mathbf{x},t) \propto \delta(\mathbf{x})$, or $\sigma(\mathbf{q},\omega) = \sigma(\omega)$ in Fourier space. However, except for very low frequencies, the ω–dependence is generally important. Generally, one calls the finite–frequency conductivity 'AC conductivity', where 'AC' stands for 'alternating current'. For $\omega \to 0$, the conductivity crosses over to the 'DC conductivity', where 'DC' stands for 'directed current'. In the DC limit, and for an isotropic medium, the field–current relation assumes the form of **Ohm's law**

$$\mathbf{j} = \sigma \mathbf{E}. \tag{4.31}$$

In the following, we wish to understand how the phenomenon of electrical conduction can be explained from our previous considerations.

There are two different ways to specify the dynamical response of a metal to external fields: We may either postulate a current–field relation in the spirit of Ohms law. *Or* we may determine a dielectric function which (in a more microscopic way) describes the response of the mobile charge carriers in the system to the presence of a field. However, since that response generally implies current flow, the simultaneous speccificiation of both a frequency dependent[8] dielectric function and a current–field relation would over–determine the problem. In the following, we discuss the two alternatives in more detail.

The phenomenological route

Let us impose Eq. (4.31) as a condition extraneous to the Maxwell equations; We simply postulate that an electric field drives a current whose temporal and spatial profile is rigidly locked to the electric field. As indiciated above, this postulate largely determines the dynamical response of the system to the external field, i.e. the dielectric function $\epsilon(\omega) = \epsilon$ has to be chosen constant. Substitution of this ansatz into the temporal Fourier transform of the Maxwell equation (1.2) obtains

$$\nabla \times \mathbf{H}(\mathbf{x},\omega) = \frac{1}{c}\left(-i\epsilon\omega + 4\pi\sigma\right)\mathbf{E}(\mathbf{x},\omega), \tag{4.32}$$

For the moment, we leave this result as it is and turn to

The microscopic route

We want to describe the electromagnetic response of a metal in terms of a model dielectric function. The dielectric function (4.23) constructed above describes the response of charge carriers harmonically bound to molecular center coordinates by a harmonic potential $\sim m_i\omega_i^2/2$ and subject to an effective damping or friction mechanism. Now, the conduction electrons of a metal mey be thought of as charge carriers whose confining potential is infinitely weak, $\omega_i = 0$, so that their motion is not tied to a reference coordinate. Still, the conduction electrons will be subject to friction mechanisms. (E.g., scattering off atomic imperfections will impede their ballistic motion through the solid.) We thus model the dielectric function of a metal as

$$\epsilon(\omega) = 1 - \frac{4\pi n_e e^2}{m\omega(\omega + \frac{i}{\tau})} + \frac{4\pi n e^2}{m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega} \overset{\omega \ll \omega_i}{\simeq} \epsilon - \frac{4\pi n_e e^2}{m\omega(\omega + \frac{i}{\tau})}, \tag{4.33}$$

[8] As we shall see below, a dielectric *constant* does not lead to current flow.

where $n_e \equiv N f_0$ is a measure of the concentration of conduction electrons, the friction coefficient of the electrons has been denoted by τ^{-1},[9] and in the last step we noted that for frequencies well below the oscillator frequencies ω_i of the valence electrons, the frequency dependence of the second contribution to the dielectric function may be neglected; We thus lump that contribution into an effective material constant ϵ. Keep in mind that the free electron contribution to the dielectric function exhaustingly describes the acceleration of charge carriers by the field (the onset of current), i.e. no extraneous current–field relations must be imposed.

Substitution of Eq. (4.33) into the Maxwell equation (1.2) obtains the result

$$\nabla \times \mathbf{H}(\mathbf{x},\omega) = -\frac{i\omega\epsilon(\omega)}{c}\mathbf{E}(\mathbf{x},\omega) = \frac{1}{c}\left(-i\omega\epsilon + \frac{4\pi n_e e^2}{m(-i\omega+\frac{1}{\tau})}\right)\mathbf{E}(\mathbf{x},\omega), \tag{4.34}$$

Comparison to the phenomenological result (4.32) yields the identification

$$\boxed{\sigma(\omega) = \frac{n_e e^2}{m(-i\omega+\frac{1}{\tau})},} \tag{4.35}$$

i.e. a formula for the **AC conductivity** in terms of electron density and mass, and the impurity collision rate.

Eq. (4.35) affords a very intuitive physical interpretations:

- ▷ For **high frequencies**, $\omega \gg \tau^{-1}$, we may approximate $\sigma \sim (i\omega)^{-1}$, or $\omega\mathbf{j} \sim \mathbf{E}$. In time space, this means $\partial_t \mathbf{j} \sim \rho\dot{\mathbf{v}} \sim \mathbf{E}$. This formula describes the ballistic acceleration of electrons by an electric field: On time scales $t \ll \tau$, which, in a Fourier sense, correspond to large frequencies $\omega \gg \tau^{-1}$ the motion of electrons is not hindered by impurity collisions and they accelarate as free particles.

- ▷ For **low frequencies**, $\omega \ll \tau^{-1}$, the conductivity may be approximated by a constant, $\sigma \sim \tau$. In this regime, the motion of the electrons is impeded by repeated impurity collisions. As a result they diffuse with a constant drift induced by the electric field.

[9] ... alluding to the fact that the attenuation of the electrons is due to collisions of impurities which take place at a rate denoted by τ.

Chapter 5

Relativistic invariance

5.1 Introduction

Consult an entry level physics textbook to re–familiarize yourself with the basic notions of special relativity!

5.1.1 Galilei invariance and its limitations

The laws of mechanics are the same in two coordinate systems K and K' moving at a constant velocity relative to one another. Within classical Newtonian mechanics, the space time coordinates $(t, \mathbf{x})$ and $(t', \mathbf{x}')$ of two such systems are related by a **Galilei transformation**,

$$\begin{aligned} t' &= t \\ \mathbf{x}' &= \mathbf{x} - \mathbf{v}t. \end{aligned}$$

Substituting this transformation into Newton's equations of motion as formulated in system K,

$$K: \qquad \frac{d^2\mathbf{x}_i}{dt^2} = -\nabla_{\mathbf{x}_i} \sum_j V_{ij}(\mathbf{x}_i - \mathbf{x}_j)$$

($\mathbf{x}_i$ are particle coordinates in system K and V_{ij} is a pair potential which, crucially, depends only on the differences between particle coordinates and not, say, on any distinguished reference point in universe.) we find that

$$K': \qquad \frac{d^2\mathbf{x}'_i}{dt'^2} = -\nabla_{\mathbf{x}'_i} \sum_j V_{ij}(\mathbf{x}'_i - \mathbf{x}'_j),$$

i.e. the equations remain form invariant. This is what is meant when we say that classical mechanics is Galilei invariant.[1]

Now, suppose a certain physical phenomenon in K (the dynamics of a water surface, say) is effectively described by a wave equation,

$$K: \qquad \left(\Delta - \frac{1}{c^2}\partial_t^2\right) f(\mathbf{x}, t) = 0.$$

[1] More precisely, Galilei invariance implies invariance under all transformations of the Galilei group, the uniform motions above, rotations of space, and translations of space.

Substitution of the Galilei transformation into this equation obtains

$$K: \qquad \left(\Delta' - \frac{1}{c^2}\partial_{t'}^2 - \frac{2}{c^2}\mathbf{v}\cdot\nabla'\partial_{t'} - \frac{1}{c^2}(\mathbf{v}\cdot\nabla')^2\right) f(\mathbf{x}', t') = 0,$$

i.e. an equation of modified form. At first sight this result may look worrysome. After all, many waves (water waves, sound waves, etc.) are of pure mechanical origin which means that wave propagation ought to be a Galilei invariant phenomenon. The resolution to this problem lies with the fact, that matter waves generally propagate in a host medium (water, say.) While in K this medium is at rest, it isn't in K'. But a wave propagating in a non–stationary medium must be controlled by a different equation of motion, i.e. there is no a priori contradiction to the principle of Galilei invariance. Equivalently, one may observe that a wave emitted by a point source stationary in K will propagate with velocity c (in K.) However, an observer in K' will observe wave fronts propagating with different velocities.[2] Mathematically, this distortion of the wave pattern is described by K''s 'wave equation'.

But what about electromagnetic waves? So far, we have never talked about a 'host medium' supporting the propagation of electromagnetic radiation. The lack of Galilean invariance of Maxwells equations then leaves us with three possibilities:

1. Nature is Galilean invariant, but Maxwells equations are incorrect (or, to put it more mildly, incomplete.)

2. An electromagnetic host medium (let's call it the **ether**) does, indeed, exist.

3. Nature is invariant under a group of space–time transformations different from Galilei. If so, Newton's equations of motion are in need of modification.

These questions became pressing in the second half of the 19th century, after Maxwell had brought the theory of electromagnetism to completion, and the exploration of all kinds of electromagnetic wave equation represented a focal point of theoretical and experimental research. In view of the sensational success of Maxwells theory the first answer was not really considered an option.

Since Newtonian mechanics wasn't a concept to be sacrificed in a light hearted manner either, the existence of an ether was postulated. However, it was soon realized that the ether medium would have to have quite eccentric physical properties. Since no one had ever actually observed an ether, this medium would have to be infinitely light and fully non–interacting with matter. (Its presumed etheric properties actually gave it its name.) Equally problematic, electromagnetic waves would uniformly propagate with the speed of light, c, only in the distinguished ether rest frame. I.e., the ether postulate was tantamount to the abandonment of the idea of 'no preferential intertial frames in nature' a concept close at heart to the development of modern physics. Towards the end of the nineteenth century, experimental evidence was mounting that the vacuum velocity of light was, independent of the observation frame, given by c; the idea of an ether became increasingly difficult to sustain.

[2] The actually measured velocity depends on the distance of the observation point to the point of origin of the wave.

5.1.2 Einsteins postulates

In essence, this summarizes the state of affairs before Einstein entered the debate. In view of the arguments outlined above, Einstein decided that 1. had to be the correct option. Specifically, he put forward two fundamental postulates

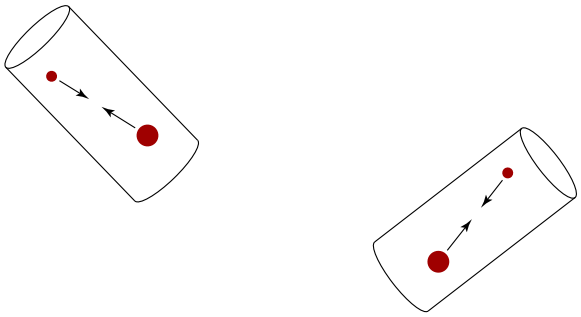


Figure 5.1: Measurement of Newtons law of gravitational attraction (any other law would be just as good) in two frames that are relatively inertial. The equations describing the force between masses will be the same in both frames

- The **postulate of relativity**: This postulate is best formulated in a negative form: there is no such thing like 'absolute rest' or an 'absolute velocity', or an 'absolute point in space'. No absolute frame exists in which the laws of nature assume a distinguished form.

  To formulate this principle in a positive form, we define two coordinate frames to be **inertial** with respect to each other if one is related to the other by a translation in space and/or time, a constant rotation, a uniform motion, or a combination of these operations. The postulate then states that the laws of nature will assume the same form (be expressed by identical fundamental equations) in both systems.

  As a corollary, this implies that physical laws must never make reference to absolute coordinates, times, angles, etc.

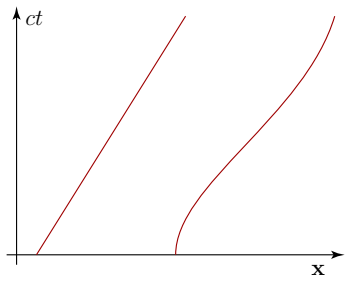
- The **postulate of the constancy of the speed of light**: the speed of light is independent of the motion of its source.

  What makes this postulate — which superficially might seem to be no more than an innocent tribute to experimental observation[3] — so revolutionary is that it implies the abandonment of 'absolute time'. For example, two events that are simultaneous in one inertial frame will in general no longer be simultaneous in other inertial frames. While, more than one hundred years later, we have grown used to its implications the revolutionary character of Einsteins second postulate cannot be exaggerated.

[3]... although the constancy of the speed of light had not yet been fully established experimentally when Einstein made his postulate.

5.1.3 First consequences

As we shall see below, the notion of an 'absolute presence' — which makes perfect sense in Newtonian mechanics[4] — becomes meaningless in Einsteins theory of relativity. When we monitor physical processes we must be careful to assign to each physical event its own space *and* time coordinate $x = (ct, \mathbf{x})$ (where the factor of c has been included for later convenience.) An event is canonical in that it can be observed in all frames (no matter whether they are relatively inertial or not). However, both its space and time coordinates are non–canonical, i.e. when observed from a different frame K' it will have coordinates $x' = (ct', \mathbf{x}')$, where t' may be different from t.



When we talk of a 'body' or a 'particle' in relativity, what we actually mean is the continuous sequence of events $x = (ct, \mathbf{x})$ defined by its instantaneous position $\mathbf{x}$ at time t (both monitored in a specific frame). The assembly of these events obtains the **world line** of the body, a (directed) curve in a space–time coordinate frame (cf. the figure for a schematic of the $(1+1)$–dimensional world lines of a body in uniform motion (left) and an accelerated body (right).)

The most important characteristic of the relative motion of inertial frames is that no acceleration is involved. This implies, in particular, that a uniform motion in one frame will stay uniform in the other; straight world lines transform into straight world lines. Parameterizing a general uniform motion in system K by $x = (ct, \mathbf{w}t + \mathbf{b})$, where $\mathbf{w}$ is the velocity and $\mathbf{b}$ an offset vector, we must require that the coordinate representation x' in any inertial system K' is again a straight line. Now, the most general family of transformations mapping straight lines onto straight lines are the **affine transformations**

$$x' = \Lambda x + a, \tag{5.1}$$

where $\Lambda : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is an arbitrary linear map from $(d+1)$–dimensional[5] space time into itself and $a \in \mathbb{R}^{d+1}$ a vector describing the displacement of the origin of coordinate systems in space and or time. For $a = 0$, we speak of a homogeneous coordinate transformation. Since any transformation may be represented as a succession of a homogeneous transformation and a trivial translation, we will focus on the former class throughout.

What conditions will the transformation matrix Λ have to fulfill in order to qualify as a transformation between inertial frames? Referring for a rigorous discussion of this question to Ref. [2], we here merely note that symmetry conditions implied by the principle of relativity nearly but not completely specify the class of permissible transformations. In particular, the class of transformations legitimate by symmetry includes the Gallilei transformations which we saw are incompatible with the physics of electromagnetic wave propagation.

At this point, **Einsteins second postulate** enters the game. Consider two inertial frames K and K' related by a homogeneous coordinate transformation. At time $t = 0$ the origins of the two systems coalesce. Let us assume, however, the K' moves with relatively to K at some constant velocity $\mathbf{v}$. Now consider the event: at time $t = 0$ a light source at $\mathbf{x} = 0$ emits a signal. In K, the wave fronts of the light signal then trace out world lines propagating at the vacuum speed of light, i.e. the spatial coordinates $\mathbf{x}$ of a wave front obey the relation

$$\mathbf{x}^2 - c^2 t^2 = 0.$$

The crucial point now is that in spite of its motion relative to the point of emanation of the light signal, an observer in K', too, will observe a light signal moving with velocity c in all directions.

[4] Two events that occur simultaneously in one frame will remain simultaneous under Galilei transformations.

[5] Although we will be foremostly interested in the case $d = 3$, it is occasionaly useful to generalize to other dimensions. Specifically, space–time diagrams are easiest to draw/imagine in $(1+1)$–dimensions.

(Notice the difference to, say, a sound wave. From K''s point of view, fronts moving in parallel to $\mathbf{v}$ would be slowed down while those moving in the opposite direction would propagate at higher speed, $\mathbf{v}'_{\rm sound} = \mathbf{v}_{\rm sound} - \mathbf{v}$.) This means that the light front coordinates in $\mathbf{x}'$ obey the same relation

$$\mathbf{x}'^2 - c^2 t'^2 = 0.$$

This additional condition unambiguously fixes the set of permissible transformations. To formulate it in the language of linear coordinate transformations (which, in view of the linearity of transformations between inertial frames is the adequate one), let us define the matrix

$$g = \{g^{\mu\nu}\} \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{5.2}$$

where all non–diagonal entries are zero. The constancy of the speed of light may then be expressed in concise form as $x^T g x = 0 \Rightarrow x'^T g x' = 0$. This condition suggests to focus on coordinate transformations for which the bilinear form $x^t g x$ is conserved,[6]

$$x^T g x = x'^T g x'. \tag{5.3}$$

Substituting $x' = \Lambda x$, we find that Λ must obey the condition

$$\Lambda^T g \Lambda = g \tag{5.4}$$

Linear coordinate transformations satisfying this condition are called **Lorentz transformations**. We thus postulate that

> All laws of nature must be invariant under affine coordinate transformations (5.1) where the homogeneous part of the transformation obeys the Lorentz condition (5.4).

While this statement 'implicitly' characterizes the set of legitimate transformations, our discussion below will require a much more explicit description. The purely mathematical task of obtaining this description will be tackled in the next section.

5.2 The mathematics of special relativity I: Lorentz group

In this section, we will develop the mathematical framework required to describe transformations between inertial frames. We will show that these transformations form a continuous group, the Lorentz group.

[6] Notice the gap in the logics of the argument. Einsteins second postulate requires Eq. (5.3) only for those space time vectors for which $x^T g x = 0$ vanishes; We now suggest to declare it as a universal condition, i.e. one that holds irrespective of the value of $x^T g x$. To show that the weaker condition implies the stronger, one may first prove (cf. Ref. [2]) that relatively straightforward symmetry considerations suffice to fix the class of allowed transformations up to one undetermined scalar constant. The value of that constant is then fixed by the 'weak condition' above. Once it has been fixed, one finds that Eq. (5.3) holds in general.

5.2.1 Background

The bilinear form introduced above defines a scalar product of $\mathbb{R}^4$:

$$g : \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}, \tag{5.5}$$

$$(x, y) \mapsto x^T g y \equiv x_\mu g^{\mu\nu} y_\nu. \tag{5.6}$$

A crucial feature of this scalar product is that it is not positive definite. For reasons to be discussed below, we call vectors whose norm is positive, $x^T g x = (ct)^2 - \mathbf{x}^2 > 0$, **time like** vectors. Vectors with negative norm will be called **space like** and vectors of vanishing norm **light like**.

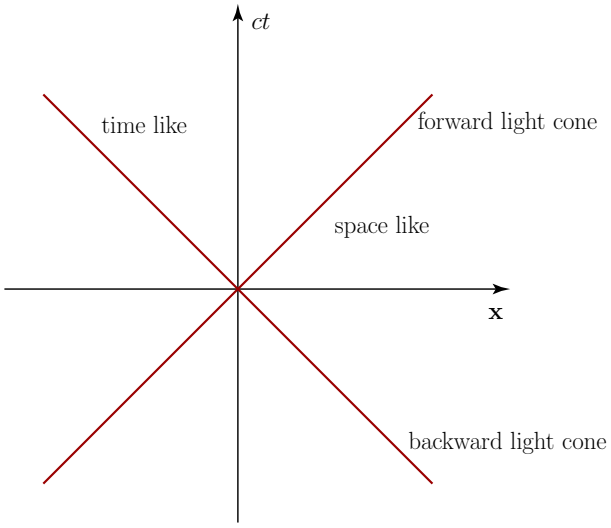


Figure 5.2: Schematic of the decomposition of space time into a time like region and a space like region. The two regions are separated by a 'light cone' of light–like vectors. (A 'cone' because in space dimensions $d > 1$ it acquires a conical structure.) The forward/backward part of the light cone extends to positive/negative times

We are interested in Lorentz transformations, i.e. linear transformations $\Lambda : \mathbb{R}^4 \rightarrow \mathbb{R}^4, x \mapsto \Lambda x$ that leave the scalar product g invariant, $\Lambda^T g \Lambda = g$. (Similarly to, say, the orthogonal transformations which leave the standard scalar product invariant.) Before exploring the structure of these transformations in detail, let us observe a number of general properties. First notice that the set of Lorentz transformations forms a group, the **Lorentz group**, L. For if Λ and Λ' are Lorentz transformations, so is the composition $\Lambda\Lambda'$. The identity transformation obviously obeys the Lorentz condition. Finally, the equation $\Lambda^T g \Lambda = g$ implies that Λ is non–singular (why?), i.e. it possesses an inverse, Λ^{-1}. We have thus shown that the Lorentz transformations form a group.

Global considerations on the Lorentz group

What more can be said about this group? Taking the determinant of the invariance relation, we find that $\det(\Lambda^T g \Lambda) = \det(\Lambda)^2 \det(g) = \det(g)$, i.e. $\det(\Lambda)^2 = 1$ which means that $\det(\Lambda) \in \{-1, 1\}$. ($\Lambda$ is a real matrix, i.e. $\det(\Lambda) \in \mathbb{R}$.) We may further observe that the absolute value of the component Λ_{00} is always larger than unity. This is shown by inspection of the 00–element of the

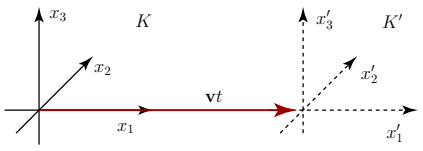
invariance relation: $1 = g_{00} = (\Lambda^T g \Lambda)_{00} = \Lambda_{00}^2 - \sum_i \Lambda_{0i}^2$. Since the subtracted term is positive (or zero), we have $|\Lambda_{00}| \geq 1$. We thus conclude that L contains four components defined by

$$\begin{array}{lll} L_+^\uparrow: & \det\Lambda = +1, & \Lambda_{00} \geq +1, \\ L_+^\downarrow: & \det\Lambda = +1, & \Lambda_{00} \leq -1, \\ L_-^\uparrow: & \det\Lambda = -1, & \Lambda_{00} > +1, \\ L_-^\downarrow: & \det\Lambda = -1, & \Lambda_{00} \leq -1. \end{array} \tag{5.7}$$

Transformations belonging to any one of these subsets cannot be continuously deformed into a transformation of a different subset (why?), i.e. the subsets are truly disjoint. In the following, we introduce a few distinguished transformations belonging to the different components of the Lorentz group.

Space reflection, or parity, $\mathcal{P} : (ct, \mathbf{x}) \to (ct, -\mathbf{x})$ is a Lorentz transformation. It belongs to the component $L_-^\uparrow$. **Time inversion**, $\mathcal{T} : (ct, \mathbf{x}) \to (-ct, \mathbf{x})$ belongs to the component $L_-^\downarrow$. The product of space reflection and time inversion, $\mathcal{PT} : (ct, \mathbf{x}) \to (-ct, -\mathbf{x})$ belongs to the component $L_+^\downarrow$. Generally, the product of two transformations belonging to a given component no longer belongs to that component, i.e. the subsets do not for sub*groups* of the Lorentz group. However, the component $L_+^\uparrow$ is exceptional in that it *is* a subgroup. It is called the **proper orthochrome Lorentz group**, or **restricted Lorentz group**. The attribute 'proper' indicates that it does not contain excpetional transformations (such as parity and time inversion) which cannot be continuosly deformed back to the identity transform. It is called 'orthochronous' because it maps the positive light cone into it self, i.e. it respects the 'direction' of time. To see that $L_+^\uparrow$ is a group, notice that it contains the identity transformation. Further, its elements can be continuously contracted to unity; however, if this property holds for two elements $\Lambda, \Lambda' \in L_+^\uparrow$, then so for the product $\Lambda\Lambda'$, i.e. the product is again in $L_+^\uparrow$.

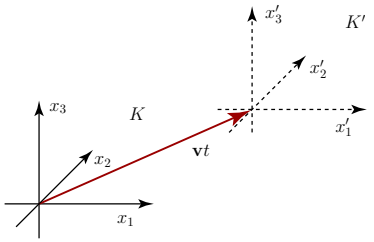
The restricted Lorentz group



The restricted Lorentz group contains those transformations which we foremostly associate with coordinate transformations between inertial frames. For example, rotations of space,

$$\Lambda_R \equiv \begin{pmatrix} 1 & \\ & R \end{pmatrix},$$

where R is a three–dimensional rotation matrix trivially fall into this subgroup. However, it also contains the second large family of homogeneous transformations between inertial frames, uniform motions.



Consider two inertial frames K and K' whose origins at time $t = 0$ coalesce (which means that the affine coordinate transformation (5.1) will be homogeneous, $a = 0$. We assume that K' moves at constant velocity $\mathbf{v}$ relative to K. Lorentz transformations of this type, $\Lambda_\mathbf{v}$, are called **special Lorentz transformations**.. Physically, they describe what is sometimes callled a **Lorentz boost**, i.e. the passage from a stationary to a moving frame. (Although we are talking about a mere coordinate transformation, i.e. no accelerated physical process is described.) We wish to explore the mathematical and physical properties of these transformations.

Without loss of generality, we assume that the vector $\mathbf{v}$ describing the motion is parallel to the $\mathbf{e}_1$ direction of K (and hence to the $\mathbf{e}'_1$–direction of K': Any general velocity vector $\mathbf{v}$ may be rotated to $\mathbf{v} \parallel \mathbf{e}_1$ by a space like rotation R. This means that a generic special Lorentz transformation $\Lambda_{\mathbf{v}}$ may be obtained from the prototypical one $\Lambda_{v\mathbf{e}_1}$ by a spatial rotation, $\Lambda_{\mathbf{v}} = \Lambda_R \Lambda_{v\mathbf{e}_1} \Lambda_R^{-1}$ (cf. the figure.)

A special transformation in $\mathbf{e}_1$–direction will leave the coordinates $x_2 = x'_2$ and $x_3 = x'_3$ unchanged.[7] The sought for transformation thus assumes the form

$$\Lambda_{v\mathbf{e}_1} = \begin{pmatrix} A_v & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

where the 2×2 matrix A_v operates in (ct, x_1)–space.

There are many ways to determine the matrix A_v. We here use a group theory inspired method which may come across as somewhat abstract but has the advantage of straightforward applicability to many other physical problems. Being element of a continuous group of transformations (a Lie group), the matrix $A_v \equiv \exp(\sum_i \lambda_i T_i)$ can be written in an exponential parameterization. Here, λ_i are real parameters and T_i two–dimensional fixed 'generator matrices' (Cf. with the conventional rotation group; the T_i's play the role of angular momentum generator matrices, and the λ_i's are generalized real–valued 'angles'.) To determine the group generators, we consider the transformation A_v for infinitesimal angles λ_i. Substituting the matrix A_v into the defining equation of the Lorentz group, using that the restriction of the metric to (ct, x) space assumes the form $g \to \sigma_3 \equiv \mathrm{diag}(1, -1)$, and expanding to first order in λ_i, we obtain

$$A_v^T \sigma_3 A_v \simeq (1 + \sum_i \lambda_i T_i^T)\sigma_3(1 + \sum_i \lambda_i T_i) \simeq \sigma_3 + \sum_i \lambda_i (T_i^T \sigma_3 + \sigma_3 T_i) \stackrel{!}{=} \sigma_3.$$

This relation must hold regardless of the value of λ_i, i.e. the matrices T_i must obey the condition $T_i^T \sigma_3 = -\sigma_3 T_i$. Up to normalization, there is only one matrix that meets this condition, viz $T_i = \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The most general form of the matrix A_v is thus given by

$$A_v = \exp\left(\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix},$$

where we denoted the single remaining parameter by λ and in the second equality expanded the exponential in a power series. We now must determine the parameter $\lambda = \lambda(v)$ in such a way that it actually describes our v–dependent transformation. Substituting A_v into the transformation law $x' = \Lambda_v x$, or

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \end{pmatrix} = A_v \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}.$$

Now, the origin of K' (having coordinate $x^{1\prime} = 0$) is at coordinate $x \stackrel{=}{} vt$. Substituting this condition into the equation above, we obtain the condition

$$\tanh\lambda = -v/c.$$

[7] To actually prove this, notice that from the point of view of an observer in K' we consider a special transformation with velocity $-v\mathbf{e}'_1$. Then apply symmetry arguments (notably the condition of parity invariance; a mirrored universe should not differ in an observable manner in its transformation behaviour from our one.) to show that any change in these coordinates would lead to a contradiction.

Using this result and introducing the (standard) notation

$$\beta = |v|/c, \qquad \gamma \equiv (1-\beta^2)^{-1/2}, \tag{5.8}$$

the special Lorentz transformation assumes the form

$$\boxed{\Lambda_{v\mathbf{e}_1} = \gamma \begin{pmatrix} 1 & -\beta & & \\ -\beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}} \tag{5.9}$$

▷ EXERCISE. Repeat the argument above for the full transformation group, i.e. not just the set of special Lorentz transformations along a certain axis. To this end, introduce an exponential representation $\Lambda = \exp(\sum_i \lambda_i T_i)$ for general elements of the restricted Lorentz group. Use the defining condition of the group to identify six linearly independent matrix generators.[8] Among the matrices T_i, identify three as the generators J_i of the spatial rotation group, $T_i = J_i$. Three others generate special Lorentz transformations along the three coordinate axes of K. (Linear combinations of these generators can be used to describe arbitrary special Lorentz transformations.)

A few more remarks on the transformations of the restricted Lorentz group:

- ▷ In the limit of small β, the transformation $\Lambda_{v\mathbf{e}_1}$ asymptotes to the Galilei transformation, $t' = t$ and $x_1' = x_1 - vt$.
- ▷ We repeat that a general special Lorentz transformation can be obtained from the transformation along $\mathbf{e}_1$ by a space–like rotation, $\Lambda_{\mathbf{v}} = \Lambda_R \Lambda_{v\mathbf{e}_1} \Lambda_{R^{-1}}$, where R is a rotation mapping $\mathbf{e}_1$ onto the unit vector in $\mathbf{v}$–direction.
- ▷ Without proof (However, the proof is by and large implied by the exercise above.) we mention that a general restricted Lorentz transformation can always be represented as

  $$\Lambda = \Lambda_R \Lambda_{\mathbf{v}},$$

  i.e. as the product of a rotation and a special Lorentz transformation.

5.3 Aspects of relativistic dynamics

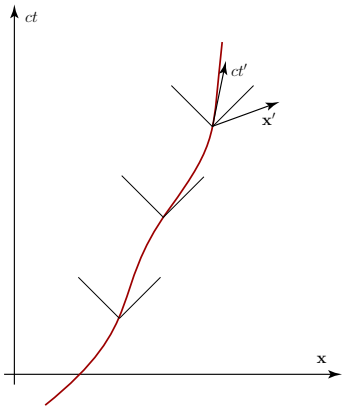
In this section we explore a few of the physical consequences deriving from Einsteins postulates; Our discussion will be embarassingly superficial, key topics relating to the theory of relativity aren't mentioned at all. Rather the prime objective of this section will be to provide the core background material required to discuss the relativistic invariance of electrodynammics below.

5.3.1 Proper time

Consider the world line of a particle moving in space in a (not necessarily uniform manner.) At any instance of time, t, the particle will be at rest in an inertial system K' whose origin coincides

[8] Matrices are elements of a certain vector space (the space of linear transformations of a given vector space.) As A set of matrices $A_1, \ldots A_k$ is linearly independent if no linear combination exists such that $\sum_i x_i A_i = 0$ vanishes.

with $\mathbf{x}(t)$ and whose instantaneous velocity w.r.t. K is given by $d_t\mathbf{x}(t)$. At this point, it is important to avoid confusion. Of course, the coordinate system whose origin globally coincides with $\mathbf{x}(t)$ is *not* inertial w.r.t. K (unless the motion is uniform.) Put differently, the origin of the inertial system K' defined by the instantaneous position $\mathbf{x}(t)$ and the instantaneous velocity $d_t\mathbf{x}(t)$ coincides with the particles position only at the very instant t. The t' axis of K' is parallel to the vector $\mathbf{x}'(t)$. Also, we know that the t' axis cannot be tilted w.r.t. the t axis by more than an angle $\pi/4$, i.e. it must stay within the instantaneous light cone attached to the reference coordinate $\mathbf{x}$.



Now, imagine an observer traveling along the world line $\mathbf{x}(t)$ and carrying a watch. At time t (corresponding to time zero in the instantaneous rest frame) the watch is reset. We wish to identify the coordinates of the event 'watch shows (infinitesimal) time $d\tau$' in both K and K'. In K' the answer to this question is formulated easily enough, the event has coordinates $(cd\tau, 0)$, where we noted that, due to the orthogonality of the space–like coordinate axes to the world line, the watch will remain at coordinate $\mathbf{x}' = 0$. To identify the K–coordinates, we first note that the origin of K' is translated w.r.t. that of K by $a = (ct, \mathbf{x}(t)$. Thus, the coordinate transformation between the two assumes the general affine form $x = Lx' + a$. In K, the event takes place at some (as yet undetermined) time dt later than t. It's spatial coordinates will be $\mathbf{x}(t) + \mathbf{v}dt$, where $\mathbf{v} = d_t\mathbf{x}$ is the instantaneous velocity of the moving observer. We thus have the identification $(cd\tau, 0) \leftrightarrow (c(t + dt), \mathbf{x} + \mathbf{v}dt)$. Comparing this with the general form of the coordinate transformation, we realize that $(cd\tau, 0)$ and $(cdt, \mathbf{v}dt)$ are related by a homogeneous Lorentz transformation. This implies, in particular, that $(cd\tau)^2 = (cdt)^2 - (\mathbf{v}dt)^2$, or

$$d\tau = dt\sqrt{1 - \mathbf{v}^2/c^2}.$$

To time between events of finite duration may be obtained by integration,

$$\tau = \int dt\sqrt{1 - \mathbf{v}^2/c^2}.$$

This equation expresses the famous phenomenon of the relativity of time: For a moving observer time proceeds slower than for an observer at rest. The time measured by the propagating watch, τ, is called **proper time**. The attribute 'proper' indicates that τ is defined with reference to a coordinate system (the instantaneous rest frame) that can be canonically defined. I.e. while the time t associated to a given point on the world line $x(t) = (ct, \mathbf{x}(t))$ will change from system to system (as we just saw), the proper time remains the same; the proper time can be used to parameterize the space–time curve in an invariant manner as $x(\tau) = (ct(\tau), \mathbf{x}(\tau))$.

5.3.2 Relativistic mechanics

Relativistic energy momentum relation

In section 5.1.2 we argued that Einstein solved the puzzle posed by the Lorentz invariance of electrodynamics by postulating that Newtonian mechanics was in need of generalization. We will begin our discussion of the physical ramifications of Lorentz invariance by developing this extended picture. Again, we start from postulates motivated by physical reasoning:

- ▷ The extended variant of Newtonian mechanics (lets call it relativistic mechanics) ought to be invariant under Lorentz transformations, i.e. the generalization of Newtons equations should assume the same form in all inertial frames.
- ▷ In the rest frame of a particle subject to mechanical forces, or in frames moving at low velocity $v/c \ll 1$ relatively to the rest frame, the equations must asymptote to Newtons equations.

Newton's equations in the rest frame K are of course given by

$$m\frac{d^2}{d\tau^2}\mathbf{x} = \mathbf{f},$$

where m is the particle's mass (in its rest frame — as we shall see, the mass is not an invariant quantity. We thus better speak of a particle **rest mass**), and $\mathbf{f}$ is the force. As always in relativity, a spatial vector (such as $\mathbf{x}$ or $\mathbf{f}$) must be interpreted as the space–like component of a four–vector. The zeroth component of x is given by $x_0 = c\tau$, so that the rest frame generalization of the Newton equation reads as $md_\tau^2 x_\mu = f_\mu$, where $f = (0, \mathbf{f})$ and we noted that for the zeroth component $md_\tau^2 x_0 = 0$, so that $f_0 = 0$. It is useful to define the **four–momentum** of the particle by $p_\mu = md_\tau x_\mu$. (Notice that both τ and the rest mass m are Lorentz invariant, i.e. the transformation behaviour of p is determined by that of x.) The zeroth component of the momentum, $md_\tau x_0 = mc$ carries the dimension [energy]/[velocity]. This suggests to define

$$p = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix},$$

where E is some characteristic energy. (In the instantaneous rest frame of the particle, $E = mc^2$ and $\mathbf{p} = 0$.)

Expressed in terms of the four–momentum, the Newton equation assumes the simple form

$$d_\tau p = f.$$

Let us explore what happens to the four momentum as we boost the particle from its rest frame to a frame moving with velocity vector $\mathbf{v} = v\mathbf{e}_1$. Qualitatively, one will expect that in the moving frame, the particle *does* carry a finite space–like momentum (which will be proportional to the velocity of the boost, v), and some renormalization of its energy. (From the point of view of the moving frame, the particle has acquired kinetic energy.)

Focusing on the 0 and 1 component of the momentum (the others won't change), Eq. (5.9) implies

$$\begin{pmatrix} mc^2/c \\ 0 \end{pmatrix} \to \begin{pmatrix} E'/c \\ p_1' \end{pmatrix} \equiv \begin{pmatrix} \gamma mc^2/c \\ -(\gamma m)v \end{pmatrix}.$$

This equation may be interpreted in two different ways Substituting the explicit formula $E = mc^2$, we find $E' = (\gamma m)c^2$ and $p_1' = -(\gamma m)v$. In Newtonian mechanics, we would have expected that a particle at rest in K carries momentum $p_1' = -mv$ in K'. The difference to the relativistic result is the renormalization of the mass factor: A particle that has mass m in its rest frame appears to carry a velocity dependent mass

$$m(\mathbf{v}) = \gamma m = \frac{m}{\sqrt{1-(v/c)^2}}$$

in K'. For $v \to c$, it becomes infinitely heavy and its further acceleration will become progressively more difficult:

Particles of finite rest frame mass cannot move faster than with the speed of light.

The energy in K' is given by $(m\gamma)c^2$, and is again affected by mass renormalization. However, a more useful representation is obtained by noting that $p^T gp$ is a conserved quantity. In K, $p^T gp = (E/c)^2 = (mc^2)^2$. Comparing with the bilinear form in K', we find $(mc)^2 = (E'/c)^2 - \mathbf{p}^2$ or

$$\boxed{E = \sqrt{(mc^2)^2 + (pc)^2}} \tag{5.10}$$

This **relativistic energy–momentum relation** determines the relation between energy and momentum of a free particle.

It most important implication is that even a free particle at rest carries energy $E = mc^2$, Einsteins world–famous result. However, this energy is usually not observable (unless it gets released in nuclear processes of mass fragmentation, with all the known consequences.) What we do observe in daily life are the changes in energy as we observe the particle in different frames. This suggests to define the **kinetic energy** of a particle by

$$T \equiv E - mc^2 = \sqrt{(mc^2)^2 + (pc)^2} - mc^2,$$

For velocities $v \ll c$, the kineatic energy $T \simeq p^2/2m$ indeed reduces to the familiar non–relativistic result. Deviations from this relation become sizeable at velocities comparable to the speed of light.

Particle subject to a Lorentz force

We next turn back to the discussion of the relativistically invariant Newton equation $d_\tau p_\mu = f_\mu$. Both, p_μ and the components of the force, f_μ, transform as vectors under Lorentz transformations. Specifically, we wish to explore this transformation behaviour on an example of obvious relevance to electrodynamics, the Newton equation of a particle of velocity $\mathbf{v}$ subject to an electric and a magnetic field. Unlike in much of our previous discussion our observer frame K is not the rest frame of the particle. The four–momentum of the particle is thus given by $p = (m\gamma c, m\gamma\mathbf{v})$. Differentiation w.r.t. the proper time of the particle, $\tau = t/\gamma$ is related to the time differentiation in K by $d_\tau = \gamma d_t$, however for notational brevity we keep differentiating w.r.t. the proper time. Finally, comparison with the familiar Lorentz force equation $d_t(m\mathbf{v}) = q(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B})$ implies that the space like components of force acting on the particle are given by, $\mathbf{f} = q\gamma(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B})$.

To complete our derivation of the Newton equation in K, we need to identify the zeroth component of the force, f_0. The zeroth component of the left hand side of the Newton equation is given by $(\gamma/c)d_t E$, i.e. $(\gamma/c)\times$ the rate at which the energy energy of the particle changes. However, we have seen before that this rate of *potential* energy change is given by $d_t U = -\mathbf{f}\cdot\mathbf{v} = -q\mathbf{E}\cdot\mathbf{v}$. Energy conservation implies that this energy gets converted into the kinetic energy, $d_t T = d_t E = +q\mathbf{E}\cdot\mathbf{v}$. This implies the identification $f_0 = (\gamma q/c)\mathbf{E}\cdot\mathbf{v}$. The generalized form of Newtons equations is thus given by

$$m d_\tau \begin{pmatrix} \gamma c \\ \gamma\mathbf{v} \end{pmatrix} = \frac{q}{c} \begin{pmatrix} \mathbf{E}\cdot(\gamma\mathbf{v}) \\ \gamma c\mathbf{E} + (\gamma\mathbf{v}) \times \mathbf{B} \end{pmatrix}.$$

The form of these equations suggests to introduce the **four–velocity** vector

$$v \equiv \begin{pmatrix} \gamma c \\ \gamma\mathbf{v} \end{pmatrix}$$

(in terms of which the four–momentum is given by $p = mv$, i.e. by multiplication by the rest mass.) The equations of motion can then be expressed as

$$m d_\tau v^\mu = F^{\mu\nu} v_\nu, \tag{5.11}$$

where we used the index raising and lowering convention originally introduced in chapter 3, i.e. $v^0 = v_0$ and $v^i = -v_i$, and the matrix F is defined by

$$\boxed{F = \{F^{\mu\nu}\} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}.} \tag{5.12}$$

The significance of Eq. (5.11) goes much beyond that of a mere reformulation of Newton's equations: The electromagnetic field enters the equation through a *matrix*, the so–called **field strength tensor**. This signals that the transformation behaviour of the electromagnetic field will be different from that of vectors. Throughout much of the course, we treated the electric field as if it was a (space–like) vector. Naively, one might have expected that in the theory of relativity, this field gets augmented by a fourth component to become a four–vector. However, a moments thought shows that this picture cannot be correct. Consider, say, a charged particle at rest in K. This particle will create a Coulomb electric field. However, in a frame K' moving relative to K, the charge will be in motion. Thus, an observer in K' will see an electric current plus the corresponding magnetic field. This tells us that under inertial transformations, electric and magnetic fields get transformed into one another. We thus need to find a relativistically invariant object accommodating the six components of the electric and magnetic field 'vectors'. Eq. (5.12) provides a tentative answer to that problem. The idea is that the fields enter the theory as a matrix and, therefore, transform in a way fundamentally different from that of vectors.

5.4 Mathematics of special relativity II: Co– and Contravariance

So far, we have focused on the Lorentz transformation behaviour of vectors in $\mathbb{R}^4$. However, our observations made in the end of the previous section suggest to include other objects (such as matrices) into our discussion. We will begin by providing some essential mathematical background and then, finally, turn to the discussion of Lorentz invariance of electrodynamics.

5.4.1 Covariant and Contravariant vectors

Generalities

Let $x = \{x^\mu\}$ be a four component object that transforms under homogeneous Lorentz transformations as $x \to \Lambda x$ or $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ in components. (By $L^\mu{}_\nu$ we denote the components of the Lorentz transformation matrix. For the up/down arrangement of indices, see the disucssion below.) Quantities of this transformation behaviour are called **contravariant vectors** (or contravariant tensors of first rank). Now, let us define another four–component object, $x_\mu \equiv g_{\mu\nu} x^\nu$. Under a Lorentz

transformation, $x_\mu \to g_{\mu\nu}\Lambda^\nu{}_\rho x^\rho = (\Lambda^{T-1})_\mu{}^\nu x_\nu$. Quantities of this transformation behaviour will be denoted as **covariant vectors** (or covariant vectors of first rank). Defining $g^{\mu\nu}$ to be the inverse of the Lorentz metric (In a basis where g is diagonal, $g = g^{-1}$), the covariant anchestor of x_μ may be obtained back by 'raising the indices' as $x^\mu = g^{\mu\nu}x_\nu$.

Critical readers may find these 'definitions' unsatisfactory. Formulated in a given basis, one may actually wonder whether they are definitions at all. For a discussion of the basis invariant meaning behind the notions of co- and contravariance, we refer to the info block below. However, we here proceed in a pragmatic way and continue to explore the consequences of the definitions (which, in fact, they are) above.

▷ Info. Consider a general real vector space V. Recall that the dual space V^* is the linear space of all mappings $\tilde{\mathbf{v}} : V \to \mathbb{R}, \mathbf{v} \mapsto \tilde{\mathbf{v}}(\mathbf{v})$. (This space is a vector space by itself which is why we denote its elements by vector–like symbols, $\tilde{\mathbf{v}}$.) For a given basis $\{\mathbf{e}_\mu\}$ of V, a dual basis $\{\tilde{\mathbf{e}}_\mu\}$ of V^* may be defined by the condition $\tilde{\mathbf{e}}_\mu(\mathbf{e}_\nu) = \delta_{\mu\nu}$. With the expansions $\mathbf{v} = \sum_\mu v^\mu \mathbf{e}_\mu$ and $\tilde{\mathbf{v}} = \sum_\mu \tilde{v}_\mu \tilde{\mathbf{e}}_\mu$, respectively, we have $\tilde{\mathbf{v}}(\mathbf{v}) = \tilde{v}_\mu v^\mu$. (Notice that components of objects in dual space will be indexed by superscripts throughout.) In the literature of relativity, elements of the vector space — then to be identified with space–time, see below — are usually called **contravariant vectors** while their partners in dual space are called **covariant vectors**.

▷ Exercise. Let $A : V \to V$ be a linear map defining a basis change in V, i.e. $\mathbf{e}_\mu = A_\mu{}^\nu \mathbf{e}'_\nu$, where $\{A_\mu{}^\nu\}$ are the matrix elements of A. Show that

- ▷ The components of a contravariant vector $\mathbf{v}$ transform as $v^\mu \to v'^\mu = (A^T)^\mu_\nu v^\nu$.
- ▷ The components of a covariant vector, $\tilde{\mathbf{w}}$ transform as $\tilde{w}_\mu \to \tilde{w}'_\mu = (A^{-1})_\mu{}^\nu \tilde{w}_\nu$.
- ▷ The action of $\tilde{\mathbf{w}}$ on $\mathbf{v}$ remains invariant, i.e. $\tilde{\mathbf{w}}(\mathbf{v}) = \tilde{w}_\mu v^\mu = \tilde{w}'_\mu v'^\nu$ does not change. (Of course, the action of the linear map $\tilde{\mathbf{w}}$ on vectors must not depend on the choice of a particular basis.)

In physics it is widespread (if somewhat tautological) practice to *define* co– or contravariant vectors by their transformation behaviour. For example, a set of d–components $\{w_\mu\}$ is denoted a covariant vector if these components change under linear transformations as $w_\mu \to A_\mu{}^\nu w_\nu$.

Now, let us assume that V is a vector space with scalar product $g : V \times V \to \mathbb{R}, (\mathbf{v}, \mathbf{w}) \mapsto v^T g w \equiv \langle v, w\rangle$. A special feature of vector spaces with scalar product is the existence of a *canonical* mapping $V \to V^*, \mathbf{v} \mapsto \tilde{\mathbf{v}}$, i.e. a mapping that to all vectors $\mathbf{v}$ canonically (without reference to a specific basis) assigns a dual element $\mathbf{v}$. The vector $\tilde{\mathbf{v}}$ is implicitly defined by the condition $\forall \mathbf{w} \in V, \tilde{\mathbf{v}}(\mathbf{w}) \stackrel{!}{=} \langle \mathbf{v}, \mathbf{w}\rangle$. For a given basis $\{\mathbf{e}_\mu\}$ of V the components of $\tilde{\mathbf{v}} = \sum_\mu \tilde{v}^\mu \tilde{\mathbf{e}}_\mu$ may be obtained as $\tilde{v}_\mu = \tilde{\mathbf{v}}(\mathbf{e}_\mu) = \langle \mathbf{v}, \mathbf{e}_\mu\rangle = v^\nu g_{\nu\mu} = g_{\mu\nu}v^\nu$, where in the last step we used the symmetry of g. With the so–called **index lowering convention**

$$v_\mu = g_{\mu\nu}v^\nu, \tag{5.13}$$

we are led to the identification $\tilde{v}_\mu = v_\mu$ and $\tilde{\mathbf{v}}(\mathbf{w}) = v_\mu w^\mu$.

Before carrying on, let us introduce some more notation: by $g^{\mu\nu} \equiv (g^{-1})_{\mu\nu}$ we denote the components of the inverse of the metric (which, in the case of the Lorentz metric in its standard diagonal representation equals the metric, but that's an exception.) Then $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$, where δ^ρ_μ is the standard Kronecker symbol. With this definition, we may introduce an **index lowering convention** as $v^\mu \equiv g^{\mu\nu}v_\nu$. Indeed, $g^{\mu\nu}v_\nu = g^{\mu\nu}g_{\nu\rho}v^\rho = \delta^\mu{}_\rho v^\rho = v^\mu$. (Notice that indices that are summed over or 'contracted' always appear crosswise, one upper and one lower index.)

Consider a map $A : V \to V, \mathbf{v} \mapsto A\mathbf{v}$. In general the action of the canonical covariant vector $\widetilde{A\mathbf{v}}$ on transformed vectors $A\mathbf{w}$ need not equal the original value $\tilde{\mathbf{v}}(\mathbf{w})$. However, it is natural to focus on those transformations that are compatible with the canonical assignment, i.e. $\widetilde{A\mathbf{v}}(A\mathbf{w}) = \tilde{\mathbf{v}}(\mathbf{w})$. In a component language, this condition translates to

$$(A^T)_\mu{}^\sigma g_{\sigma\rho} A^\rho{}_\nu = g_{\mu\nu}$$

or $A^T g A = g$. I.e. transformations compatible with the canonical identification (vector space) $\leftrightarrow$ (dual space.) have to respect the metric. E.g., in the case of the Lorentz metrix, the 'good' transformations belong to the Lorentz group.

Summarizing, we have seen that the invariant meaning behind contra– and covariant vectors is that of vectors and dual vectors resepectively. Under Lorentz transformations these objects behave as is required by the component definitions given in the main text. A general tensor of degree (n, m) is an element of $(\otimes_1^n V) \otimes (\otimes_1^m V^*)$.

The definitions above may be extended to objects of higher complexity. A two component quantity $W^{\mu\nu}$ is called a contravariant **tensor of second degree** if it transforms under Lorentz transformations as $W^{\mu\nu} \to \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} W^{\mu'\nu'}$. Similiarly, a covariant tensor of second degree transforms as $W_{\mu\nu} \to (\Lambda^{T-1})_\mu{}^{\mu'} (\Lambda^{T-1})_\nu{}^{\nu'} W_{\mu'\nu'}$. Covariant and contravariant tensors are related to each other by index raising/lowering, e.g. $W_{\mu\nu} = g_{\mu\mu'} g_{\nu\nu'} W^{\mu'\nu'}$. A mixed second rank tensor $W_\mu{}^\nu$ transforms as $W_\mu{}^\nu \to (\Lambda^{T-1})_\mu{}^{\mu'} \Lambda^\nu{}_{\nu'} W_{\mu'}{}^{\nu'}$. The generalization of these definitions to tensors of higher degree should be obvious.

Finally, we define a contravariant **vector field** (as opposed to a fixed vector) as a field $v_\mu(x)$ that transforms under Lorentz transformations as $v^\mu(x) \to v'^\mu(x') = \Lambda^\mu{}_\nu v^\nu(x)$. A **Lorentz scalar** (field), or tensor of degree zero is one that does not actively transform under Lorentz transformations: $\phi(x) \to \phi'(x') = \phi(x)$. Covariant vector fields, tensor fields, etc. are defined in an analogous manner.

The summation over a contravariant and a covariant index is called a **contraction**. For example, $v^\mu w_\mu \equiv \phi$ is the contraction of a contravariant and a covariant vector (field). As a result, we obtain a Lorentz scalar, i.e. an object that does not change under Lorentz transformations. Generally, the contraction of two indices lowers the tensor degree of an object by two. For example the contraction of a mixed tensor of degree two and a contravariant tensor of degree one, $A^\mu{}_\nu v^\nu \equiv w^\mu$ obtains a contravariant tensor of degree one, etc.

5.4.2 The relativistic covariance of electrodynamics

We now have everything in store to prove the relativistic covariance of electrodynamics, i.e. the form–invariance of its basic equations under Lorentz transformations. Basically, what we need to do is assign a definite transformation behaviour scalar, co–/contravariant tensor, etc. to the building blocks of electrodynamics.

The coordinate vector x^μ is a contravariant vector. In electrodynamics we frequently differentiate w.r.t. the compoents x^μ, i.e. the next question to ask is whether the four–dimensional gradient $\{\partial_{x^\mu}\}$ is co– or contravariant. According to the chain rule,

$$\frac{\partial}{\partial x'^{\mu'}} = \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial}{\partial x^\mu}.$$

Using that $x^\mu = (\Lambda^{-1})^\mu{}_{\mu'} x'^{\mu'}$, we find that

$$\frac{\partial}{\partial x'^{\mu'}} = (\Lambda^{-1T})_{\mu'}{}^{\mu} \frac{\partial}{\partial x^\mu},$$

i.e. $\partial_{x^\mu} \equiv \partial_\mu$ transforms as a covariant vector. In components,

$$\partial_\mu = (c^{-1}\partial_t, \nabla), \qquad \partial^\mu = (c^{-1}\partial_t, -\nabla).$$

The four–current vector

We begin by showing that the four–component object $j = (c\rho, \mathbf{j})$ is a Lorentz vector. Consider the current density carried by a point particle at coordinate $\mathbf{x}(t)$ in some coordinate system K. The four–current density carried by the particle is given by $j(x) = q(c, d_t\mathbf{x}(t))\delta(\mathbf{x} - \mathbf{x}(t))$, or $j^\mu(x) = q d_t x^\mu(t)\delta(\mathbf{x} - \mathbf{x}(t))$, where $x^\mu(t) = (ct, \mathbf{x}(t))$. To show that this is a contravariant vector, we introduce a dummy integration,

$$j^\mu(x) = q \int d\tilde{t}\, \frac{dx^\mu(\tilde{t})}{d\tilde{t}} \delta(\mathbf{x} - \mathbf{x}(\tilde{t}))\delta(t - \tilde{t}) = qc \int d\tau\, \frac{dx^\mu}{d\tau} \delta^4(x - x(\tau)),$$

where in the last step we switched to an integration over the proper time $\tau = \tau(\tilde{t})$ uniquely assigned to the world line parameterization $x(\tilde{t}) = (c\tilde{t}, \mathbf{x}(\tilde{t}))$ in K. Now, the four–component δ–distribution, $\delta^4(x) = \delta(\mathbf{x})\delta(ct)$ is a Lorentz scalar (why?). The proper time, τ, also is a Lorentz scalar. Thus, the transformation behaviour of j^μ is dictated by that of x^μ, i.e. j^μ defines a contravariant vector.

As an important corrolary we note that the **continuity equation** — the contraction of the covariant vector ∂_μ and the contravariant vector j^μ is Lorentz invariant, $\partial_\mu j^\mu = 0$ is a Lorentz scalar.

Electric field strength tensor and vector potential

In section 5.3.2 we obtained Eq. (5.11) for the Lorentz invariant generalization of Newton's equations. Since v_μ and v^ν transform a covariant and contravariant vectors, respectively, the matrix $F^{\mu\nu}$ must transform as a contravariant tensor of rank two. Now, let us define the four **vector potential** as $\{A^\mu\} = (\phi, \mathbf{A})$. It is then a straightforward exercise to verify that the field strength tensor is obtained from the vector potential as

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.} \tag{5.14}$$

(Just work out the antisymmetric combination of derivatives and compare to the definitions $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\nabla\phi - c^{-1}\partial_t \mathbf{A}$.) This implies that the four–component object A^μ indeed transforms as a contravariant vector.

Now, we have seen in chapter 3.2 that the **Lorentz condition** can be expressed as $\partial_\mu A^\mu = 0$, i.e. in a Lorentz invariant form. In the Lorentz gauge, the inhomogeneous wave equations assume the form

$$\Box A^\mu = \frac{4\pi}{c} j^\mu, \tag{5.15}$$

where $\Box = \partial_\mu \partial^\mu$ is the Lorentz invariant wave operator. Formally, this completes the proof of the Lorentz invariance of the theory. The combination Lorentz–condition/wave equations, which we saw

carries the same information as the Maxwell equations, has been proven to be invariant. However, to stay in closer contact to the original formulation of the theory, we next express Maxwells equations themselves in a manifestly invariant form.

Invariant formulation of Maxwells equations

One may verify by direct inspection that the two inhomogeneous Maxwell equations can be formulated in a covariant manner as

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\mu.} \tag{5.16}$$

To obtain the covariant formulation of the homogeneous equations a bit of preparatory work is required: Let us define the fourth rank antisymmetric tensor as

$$\epsilon^{\mu\nu\rho\sigma} \equiv \begin{cases} 1, & (\mu,\nu,\rho,\sigma) = (0,1,2,3) \text{or an even permutation} \\ -1 & \text{for an odd permutation} \\ 0 & \text{else.} \end{cases} \tag{5.17}$$

One may show (do it!) that $\epsilon^{\mu\nu\rho\sigma}$ transforms as a contravariant tensor of rank four (under the transformations of the unit–determinant subgroup L_+.) Contracting this object with the covariant field strength tensor $F_{\mu\nu}$, we obtain a contravariant tensor of rank two,

$$\mathcal{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},$$

known as the **dual field strength tensor**. Using (5.18), it is straightforward to verify that

$$\boxed{\mathcal{F} = \{\mathcal{F}^{\mu\nu}\} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix},} \tag{5.18}$$

i.e. that $\mathcal{F}$ is obtained from F by replacement $\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$. One also verifies that the homogeneous equations assume the covariant form

$$\boxed{\partial_\mu \mathcal{F}^{\mu\nu} = 0.} \tag{5.19}$$

This completes the invariance proof. Critical readers may find the definition of the dual tensor somewhat un–motivated. Also, the excessive use of indices — something one normally tries to avoid in physics — does not help to make the structure of the theory transparent. Indeed, there exists a much more satisfactory, coordinate independent formulation of the theory in terms of differential forms. However, as we do not assume familiarity of the reader with the theory of forms, all we can do is encourage him/her to learn it and to advance to the truly invariant formulation of electrodynamics not discussed in this text ...

Chapter 6

Appendix

6.1 Vector calculus in curvelinear coordinates

In analytical electrodynamics we often think about problems with an exceptionally high degree of symmetry — rotational symmetry, axial symmetry, etc. These problems are conveniently formulated in non–cartesian coordinates; To solve them we need to express all analytical operations of vector calculus in such coordinates.

6.1.1 Curvelinear orthogonal systems

Focusing on the case of three dimensional space, let $\mathbf{r}(u_1, u_2, u_3)$ be the parameterization of a point in space in terms of three coordinates u_i. (By way of example, you may think of spherical coordinates, $u_1 = r, u_2 = \theta, u_3 = \phi$. Consider the three vectors $\mathbf{v}_i \equiv \partial_{u_i}\mathbf{r}(u)$. In general, these vectors will not be normalized, so let us consider their normalized descendants $\mathbf{e}_i \equiv g_i^{-1}\mathbf{v}_i$, where $g_i \equiv |\mathbf{v}_i|$. For any point $\mathbf{r}$, the three vectors $\{\mathbf{e}_i\}$ form a system of coordinate vectors pointing in the direction of the coordinate lines of the coordinate system, i.e. those lines along which two of the coordinates are kept constant (see the figure.)

▷ Example: Specifically, for spherical coordinates, $\mathbf{r} = (r\sin\theta\cos\phi, r\sin\theta\cos\phi, r\cos\theta)^T$. It is then straightforward to verify that $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ form the familiar triple of spherical coordinate vectors and that the normalization factors are given by $g_r = 1$, $g_\theta = r$, $g_\phi = r\sin\theta$.

Now, most coordinate systems of practical relevance are **orthogonal** in the sense that $\langle \mathbf{e}_i, \mathbf{e}_j\rangle = \delta_{ij}$, i.e. the coordinate vectors form an orthonormal system. An arbitrary vector field $\mathbf{v}$ can be expanded in this coordinate system as $\mathbf{v}(\mathbf{r}) = \sum_i v_i(\mathbf{r})\mathbf{e}_i(\mathbf{r})$, where $v_i(\mathbf{r}) = \langle \mathbf{v}(\mathbf{r}), \mathbf{e}_i(\mathbf{r})\rangle$ and the notation indicates that both the coefficient functions and the coordinate vectors may depend on $\mathbf{r}$.

6.1.2 Vector differential operations in curvelinear coordinates

Gradient

Let $f(\mathbf{r})$ be a differentiable function. The gradient of f, ∇f is a vector that points in the direction of maximal variation of f. Put differently, the of overlap of ∇f with an arbitrary reference vector

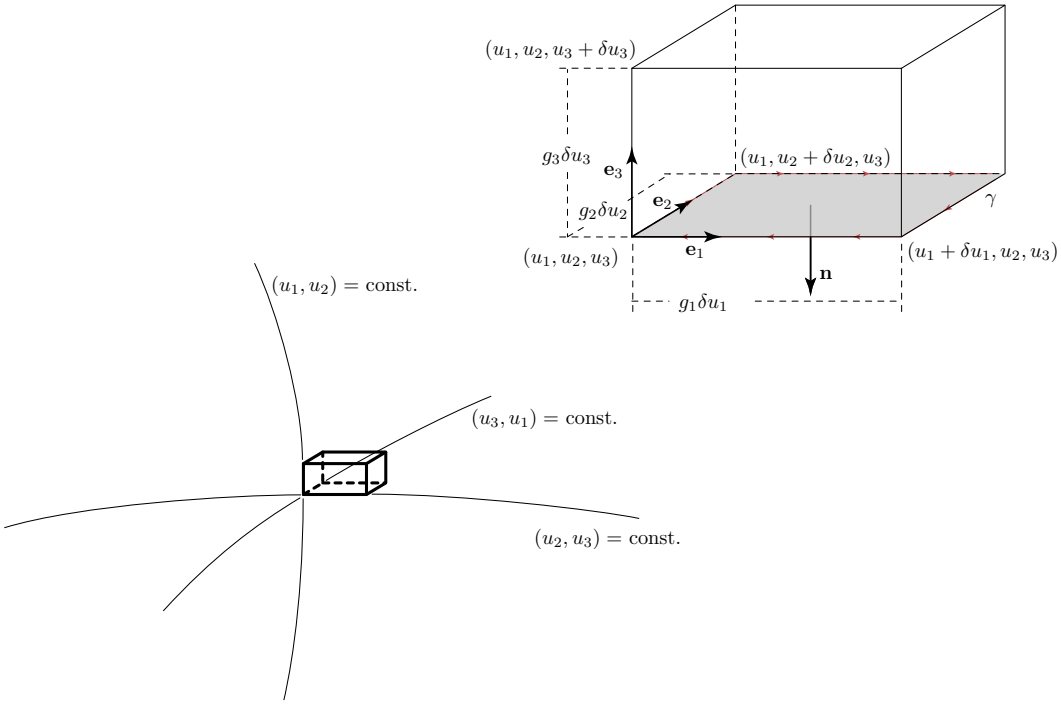


Figure 6.1: On the definition of an infinitesimal reference volume in an orthogonal curvelinear coordinate system.

$\mathbf{v}$ equals the directional derivative of f with respect to $\mathbf{v}$:

$$\forall \mathbf{v} : \ \langle \nabla f, \mathbf{v} \rangle \equiv \partial_{\mathbf{v}} f. \tag{6.1}$$

We now use this definition to compute the components of ∇f in an arbitrary orthogonal coordinate system: $\nabla f_i \equiv \langle \nabla f, \mathbf{e}_i \rangle = \partial_{\mathbf{e}_i} f = (g_i)^{-1} \partial_{g_i \mathbf{e}_i} f = (g_i)^{-1} \partial_{\mathbf{v}_i} f$. Now, recall that to compute a directional derivative $\partial_{\mathbf{v}} f$, we need a curve $\boldsymbol{\gamma}(s)$ whose velocity at $s = u$, equals $\mathbf{v}$: $d_u|_{u=u_0} \boldsymbol{\gamma}(u) = \mathbf{v}(\boldsymbol{\gamma}(u_0))$. Then, $\partial_{\mathbf{v}} f = d_u|_{u=u_0} f(\gamma(u))$. Presently, a curve that does the job is $\mathbf{r}(u_1, u_2, u_3)$: by definition $\partial_{u_i} \mathbf{r} = \mathbf{v}_i$, i.e. $\mathbf{r}(u_1, u_2, u_3)$ is a suitable parameter curve and $\partial_{\mathbf{v}_i} f = \partial_{u_i} f$. Summarizing we have found that $\nabla f_i = g_i \partial_{u_i} f$, or

$$\boxed{\nabla f = \sum_i \frac{1}{g_i} (\partial_{u_i} f)\, \mathbf{e}_i.} \tag{6.2}$$

▷ Example: For **spherical coordinates**,

$$\nabla f = \partial_r f + \frac{1}{r} \partial_\theta f + \frac{1}{r \sin\theta} \partial_\phi f.$$

Divergence

To obtain generalizations of the cartesian variant $\nabla \cdot \mathbf{v} = \partial_{x_i} \mathbf{v}_i$ of the divergence, we first need to give this vector differentiation operation a coordinate–independent meaning. Indeed, the 'true' meaning of the divergence — a measure of the source contents of a vector field — is expressed by Gauß's theorem (1.23). Specifically, for an infinitesimally small reference volume Gauß's theorem assumes the form

$$\nabla \cdot \mathbf{v} = \frac{1}{V(S)} \int_S d\sigma \, \mathbf{v} \cdot \mathbf{n},$$

where $V(S)$ denotes the volume of the area enclosed by S and we have assumed continuity of $\mathbf{v}$, i.e. $\int_{V(S)} d^3x \, \nabla \cdot \mathbf{v} \simeq V(S) \nabla \cdot \mathbf{v}$. Now, let us assume that $V(S)$ is the cuboid spanned by the three vectors $g_i \mathbf{e}_i \delta u_i$, where the three parameters δu_i are infinitesimal (see Fig. 6.1.) Notice that these vectors describe the change of the position vector $\mathbf{r}$ under changes of the coordinates, e.g. $\mathbf{r}(u_1, u_2, u_3 + \delta u_3) - \mathbf{r}(u_1, u_2, u_3) \simeq \partial_{u_3} \mathbf{r} \delta u_3 = g_3 \mathbf{e}_3 \delta u_3$. Clearly, the volume of $V(S)$ (also denoted by $V(S)$ for brevity) is given by $V(S) = \prod_i g_i \delta u_i$. The integral of $\mathbf{v}$ over the surface S obtains by adding the product of the areas of the faces of $V(S)$ times the components of $\mathbf{v}$ normal to these faces. For example, the area of the bottom face (cf. Fig. 6.1) is given by $\delta u_1 \delta u_2 \, (g_1 g_2)(u_1, u_2, u_3)$, where the argument (u_1, u_2, u_3) indicates that the metric factors g_i depend on the coordinate data. The unit vector normal to that surface is given by $-\mathbf{e}_3$, i.e. the normal component of $\mathbf{v}$ is $-v_3(u_1, u_2, u_3)$, where v_i are the components in the coordinate system $\{\mathbf{e}_i\}$. The opposite (top) face of the box has an area $\delta u_1 \delta u_2 \, (g_1 g_2)(u_1, u_2, u_3 + \delta u_3)$, and the normal component of $\mathbf{v}$ is given by $+v_3(u_1, u_2, u_3 + \delta u_3)$. Adding the contribution of the top and the bottom face to the surface integral, we obtain $\delta u_1 \delta u_2 [(g_1 g_2 v_3)(u_1, u_2, u_3 + \delta u_3) - (g_1 g_2 v_3)(u_1, u_2, u_3)] \simeq \delta u_1 \delta u_2 \delta u_3 \, \partial_{u_3}(g_1 g_2 v_3)$. Adding the contribution of the other faces and dividing by $V(S)$, we thus obtain

$$\boxed{\nabla \cdot \mathbf{v} = \frac{1}{g_1 g_2 g_3} \left[\partial_{u_1}(g_2 g_3 v_1) + \partial_{u_2}(g_3 g_1 v_2) + \partial_{u_3}(g_1 g_2 v_3) \right].} \tag{6.3}$$

For cartesian coordinates this expression reduces to the familiar form given above.

▷ Example: For **spherical coordinates**,

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2} \partial_r (r^2 v_r) + \frac{1}{r} \partial_\theta (v_\theta) + \frac{1}{r \sin\theta} \partial_\phi (\sin\theta \, v_\phi),$$

where $v_r = \mathbf{v} \cdot \mathbf{e}_r$, etc.

Curl

As with Gauß's theorem above, we consider an infinitesimal variant of Stoke's theorem:

$$\mathbf{v} \times \mathbf{n} = \frac{1}{S} \int_\gamma d\mathbf{s} \cdot \mathbf{v},$$

where S is the area of an infinitesimal surface element (also denoted by S) bounded by a curve γ and $\mathbf{n}$ is the normal unit vector to S. Specifically, let us consider the example S=(bottom face of the cuboid volume above). Then $S = \delta u_1 \delta u_2 \, g_1 g_2$ and $\mathbf{n} = -\mathbf{e}_3$. Keeping in mind that the

orientation of the contour γ must be chosen so as to conform with the 'right hand rule', the line integral around the rectangular contour surrounding S evaluates to (see the figure)

$$\begin{aligned}\int_\gamma d\mathbf{s}\cdot\mathbf{v} \simeq \delta u_2(\mathbf{v}\cdot\mathbf{e}_2 g_2)(u_1,u_2,u_3) + \delta u_1(\mathbf{v}\cdot\mathbf{e}_1 g_1)(u_1,u_2+\delta u_2,u_3) - \\ -\delta u_2(\mathbf{v}\cdot\mathbf{e}_2 g_2)(u_1+\delta u_1,u_2,u_3) - \delta u_1(\mathbf{v}\cdot\mathbf{e}_1 g_1)(u_1+\delta u_1,u_2,u_3) \simeq \\ \simeq -\delta u_1\delta u_2(\partial_{u_1}(g_2 v_2) - \partial_{u_2}(g_1 v_1)),\end{aligned}$$

where $\mathbf{v}_i = \mathbf{v}\cdot\mathbf{e}_i$ are the components of $\mathbf{v}$ in the coordinate system and the coordinate reference points are evaluated at the corners of the rectangle sitting at the beginning of the outgoing line–stretches. (To first order in the parameters δu_i, the choice of the coordinate convention is immaterial, provided one sticks to a consistent rule.) Substituting this result into the r.h.s. of the infinitesimal variant of Stokes theorem, we obtain $(\nabla\times\mathbf{v})_3 = (g_1 g_2)^{-1}(\partial_{u_1}(g_2 v_2) - \partial_{u_2}(g_1 v_1))$. Analogous results are obtained for the two other components of the curl. Expressed in terms of the Levi–Civita symbol,

$$\boxed{(\nabla\times\mathbf{v})_i = \sum_{jk}\frac{\epsilon_{ijk}}{g_j g_k}\partial_{u_j}(g_k v_k).} \tag{6.4}$$

For Cartesian coordinates we rediscover the standard formula $(\nabla\times\mathbf{v})_i = \epsilon_{ijk}\partial_j v_k$.

▷ Example: For **spherical coordinates**,

$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta v_\phi) - \partial_\phi v_\theta\right]\mathbf{e}_r + \left[\frac{1}{r\sin\theta}\partial_\phi v_r - \frac{1}{r}\partial_r(r v_\phi)\right]\mathbf{e}_\theta + \frac{1}{r}\left[\partial_r(r v_\theta) - \partial_\theta v_r\right]\mathbf{e}_\phi.$$

Laplace operator

A generalized expression for the Laplace operator $\Delta = \nabla\cdot\nabla$ is obtained by substituting Eq.(6.2) into Eq.(6.3):

$$\boxed{\Delta = \frac{1}{g_1 g_2 g_3}\left[\partial_{u_1}\frac{g_2 g_3}{g_1}\partial_{u_1} + \partial_{u_2}\frac{g_3 g_1}{g_2}\partial_{u_2} + \partial_{u_3}\frac{g_1 g_2}{g_3}\partial_{u_3}\right].} \tag{6.5}$$

▷ Example: For **spherical coordinates**,

$$\Delta = \frac{1}{r^2}\partial_r r^2\partial_r + \frac{1}{r^2\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\phi^2.$$

For the convenience of the reader, the basic operations of vector calculus in spherical and cylindrical coordinates are summarized in the table below.

Cylindrical	$\nabla f = (\partial_\rho f)\mathbf{e}_\rho + \rho^{-1}(\partial_\phi f)\mathbf{e}_\phi + (\partial_z f)\mathbf{e}_z,$
(ρ, ϕ, z)	$\nabla \cdot \mathbf{v} = \rho^{-1}\partial_\rho(\rho v_\rho) + \rho^{-1}\partial_\phi v_\phi + \partial_z v_z,$
	$\nabla \times \mathbf{v} = \left[\rho^{-1}\partial_\phi v_z - \partial_z v_\phi\right]\mathbf{e}_\rho + \left[\partial_z v_\rho - \partial_\rho v_z\right]\mathbf{e}_\phi + \rho^{-1}\left[\partial_\rho(\rho v_\phi) - \partial_\phi v_\rho\right]\mathbf{e}_z,$
	$\Delta f = \rho^{-1}\partial_\rho\rho\partial_\rho f + \rho^{-2}\partial_\phi^2 f + \partial_z^2\phi.$
Spherical	$\nabla f = \partial_r f + r^{-1}\partial_\theta f + (r\sin\theta)^{-1}\partial_\phi f,$
(ρ, ϕ, z)	$(\nabla \cdot \mathbf{v}) = r^{-2}\partial_r(r^2 v_r) + (r\sin\theta)^{-1}\partial_\theta(\sin\theta v_\theta) + (r\sin\theta)^{-1}\partial_\phi v_\phi,$
	$\nabla \times \mathbf{v} = (r\sin\theta)^{-1}\left[\partial_\theta(\sin\theta v_\phi) - \partial_\phi v_\theta\right]\mathbf{e}_r + \left[(r\sin\theta)^{-1}\partial_\phi v_r - r^{-1}\partial_r(r v_\phi)\right]\mathbf{e}_\theta +$ $+ r^{-1}\left[\partial_r(r v_\theta) - \partial_\theta v_r\right]\mathbf{e}_\phi.$
	$\Delta f = r^{-2}\partial_r r^2\partial_r f + (r^2\sin\theta)^{-1}\partial_\theta\sin\theta\partial_\theta f + (r^2\sin^2\theta)^{-1}\partial_\phi^2 f.$

Table 6.1: The basic operations of vector calculus in spherical and in cylindrical coordinates

Bibliography

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