Spin and the Addition of Angular Momentum Using Tensor Notation

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Spin is perhaps one of the most confusing concepts in introductory Quantum Mechanics, because it has no classical analog. The purpose of these notes is to restate and expand upon the material found in *Griffiths*, using a much more careful, and hopefully much more conceptually revealing, notation than he uses.

1 The Basics

The theory of angular momentum, both orbital and spin, is built up around a set of operators* S_x, S_y, and S_z that have commutation relations given by

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y. \tag{1}$$

Out of these operators, we can construct S^2, which is given by

$$S^2 = S_x^2 + S_y^2 + S_z^2, \tag{2}$$

and which commutes with each of S_x, S_y, and S_z.

Because S^2 and S_z commute, we can construct simultaneous eigenstates of these operators. We denote these by $|s\ m_s\rangle$, where s is an integer or half-integer greater than or equal to zero and m_s takes on all values between s and $-s$ (inclusive) in integer steps†. The relevant eigenvalue equations are

$$S^2 |s\ m_s\rangle = s(s+1)\hbar^2 |s\ m_s\rangle \tag{3a}$$

$$S_z |s\ m_s\rangle = m_s\hbar |s\ m_s\rangle \tag{3b}$$

*For simplicity, I'll stick to talking about spin for the majority of these notes, but remember that everything I say also holds true for orbital angular momentum, with the exception that there is no such thing as half-integer orbital angular momentum.

†That means that for any given s, there are $2s+1$ possible values for m_s.

The simplest non-trivial example of a spin system is spin-$\frac{1}{2}$. The basis of eigenstates for this system is given by $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$. In matrix notation, these basis states are given by

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4}$$

and the matrices mentioned above are given by

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

and

$$S^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

2 The "Separate" Representation

Suppose we have two particles, one of spin s_1 and the other of spin s_2. We would like to describe the possible spin states that these particles can be in. It turns out that there are two good bases for doing so, one that treats the spin of each particle separately and one that looks at the combined spin of the system as a whole.

Let's study the former first. The particle of spin s_1 can be in one of $2s_1 + 1$ different spin states, one for each possible value of m_s for that particle. Likewise, the particle of spin s_2 can be in one of $2s_2+1$ different spin states. If we treat each particle independently, that means that there are $2s_1+1$ times $2s_2+1$ different possible states for the two particles to be in[‡]. We can denote these states by

$$|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle\,. \tag{7}$$

This notation emphasizes the fact that we are thinking about our states in terms of the eigenstates of the spin operators for each particle separately. The $\otimes$ represents an operation called a *tensor product*[§]. In essence, the tensor product indicates that we

[‡]This is true for the same reason that if you have 3 shirts and 2 pairs of pants you can make 2×3=6 outfits. It's just basic combinatorics.

[§]The concept of a tensor product and a direct sum (which will come up later in these notes) ultimately arise from the fact that the theory of angular momentum in quantum mechanics is an application of group theory. I will try to borrow useful notation and concepts from group theory in these notes without actually going into all of the mathematical formalism, as that's another course's worth of material right there.

are forming a new Hilbert space for our particles' spins as a product of each individual particle's original Hilbert space.

This is abstract, so let's investigate what it means with some calculations. First of all, the statement that our two particles have their own Hilbert spaces implies that there are two distinct sets of spin operators, one set for each particle, that we will denote by $S^{2\,(1)}$ and $S_i^{(1)}$ for particle 1[¶] and $S^{2\,(2)}$ and $S_i^{(2)}$ for particle 2. The (1) operators *only* act on the first ket in our tensor product state, and the (2) operators *only* act on the second one. For example,

$$S_z^{(1)} \left|1\ 1\right\rangle \otimes \left|3\ -2\right\rangle = \hbar \left|1\ 1\right\rangle \otimes \left|2\ -2\right\rangle , \tag{8}$$

but

$$S_z^{(2)} \left|1\ 1\right\rangle \otimes \left|3\ -2\right\rangle = -2\hbar \left|1\ 1\right\rangle \otimes \left|2\ -2\right\rangle . \tag{9}$$

It's possible to be even more precise in our notation. Since the tensor product allows us to express the states that we get when we combine two Hilbert spaces, it should also allow us to express the operators we get when we combine those Hilbert spaces. In particular, we can write the operators from the preceding paragraph as

$$S^{2\,(1)} = S^2 \otimes \mathbb{1} \tag{10a}$$

$$S^{2\,(2)} = \mathbb{1} \otimes S^2, \tag{10b}$$

and similar for the S_i operators[‖]. The benefit of this notation is that there is absolutely no ambiguity about which part of the combined Hilbert space out operator is acting upon: operators to the left of the tensor product symbol act on kets to the left and operators on the right act on kets to the right. For instance,

$$(A \otimes B)(C \otimes D)\left(\left|\phi\right\rangle \otimes \left|\psi\right\rangle\right) = AC\left|\phi\right\rangle \otimes BD\left|\psi\right\rangle . \tag{11}$$

The other big benefit of this notation is that we can write all sorts of "mixed" operators, like $S_x \otimes S^2$, which would have been more difficult to write in the (1) (2) notation. To make all this a bit clearer, let's do an explicit example:

$$\begin{aligned} (S^2 \otimes S_z)\left(\left|2\ -1\right\rangle \otimes \left|1\ 1\right\rangle\right) &= S^2\left|2\ -1\right\rangle \otimes S_z\left|1\ 1\right\rangle \\ &= 2(2+1)\hbar^2\left|2\ 1\right\rangle \otimes \hbar\left|1\ 1\right\rangle \\ &= 6\hbar^3\left(\left|2\ -1\right\rangle \otimes \left|1\ 1\right\rangle\right) . \end{aligned} \tag{12}$$

[¶]Whenever you see a subscript i, take it to means any of x, y, or z. It will save me much typing if I don't have to write down all three operators whenever I make a statement that applies to any of them.

[‖]Here the symbol $\mathbb{1}$ stands for the identity matrix; its the operator that does nothing at all to a ket.

Finally, we'd like to know what the bras that correspond to the kets in our combined Hilbert space look like. Remembering that bras and kets are Hermitian conjugates of each other, it's not too hard to believe that

$$(|\phi\rangle \otimes |\psi\rangle)^\dagger = \langle\phi| \otimes \langle\psi| \,. \tag{13}$$

With this we can also express inner products and expectation values:

$$(\langle\lambda| \otimes \langle\xi|)\,(A \otimes B)\,(|\phi\rangle \otimes |\psi\rangle) = \langle\lambda|A|\phi\rangle \otimes \langle\xi|B|\psi\rangle = \langle\lambda|A|\phi\rangle\,\langle\xi|B|\psi\rangle\,, \tag{14}$$

where in the last equality we used the fact that inner products are just numbers, so the tensor product should just reduce to regular multiplication.

2.1 More Operator Examples: Total Spin

One useful quantity that we might want to calculate for our system of two particles is the total spin of the system (both the magnitude of the spin and the component in the z-direction). Since these are observables, we should be able to write them as operators in our product Hilbert space.

First, let's think about how we might write $S_{z,total}$. For the state $|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle$, the z-component of particle 1's spin has a value of $m_{s_1}\hbar$ and the z-component of particle 2's spin has a value of $m_{s_2}\hbar$. Therefore, the total spin in the z-direction must be the sum of those, $(m_{s_1} + m_{s_2})\hbar$. This suggests that we write $S_{z,total}$ as**

$$S_{z,total} = S_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z, \tag{15}$$

**Note that this is the "correct" way to write one component of Griffith's equation 4.176. As written in the textbook, 4.176 make little sense because (1) S is not a well-defined operator, unlike S^2 and S_i and (2) it makes no sense to add operators that live in different Hilbert spaces. Of course, what's really going on here is that Griffith's equation is shorthand for Eq. (15) and the corresponding equations for $S_{x,total}$ and $S_{y,total}$. From speaking with some of you, I believe that much of the confusion you all are experiencing about addition of angular momentum comes from the ill-defined notation in the text and the belief that sparing you the details of the mathematical formalism will make the material more comprehensible. I think that's nonsense; hence, I wrote these notes.

because when we act it on our generic state we get

$$\begin{aligned} S_{z,total}(|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle) &= (S_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z)(|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle) \\ &= (S_z \otimes \mathbb{1})(|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle) + (\mathbb{1} \otimes S_z)(|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle) \\ &= (S_z\,|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle) + (|s_1\ m_{s_1}\rangle \otimes S_z\,|s_2\ m_{s_2}\rangle) \\ &= m_{s_1}\hbar(|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle) + m_{s_2}\hbar(|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle) \\ &= (m_{s_1} + m_{s_2})\hbar(|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle), \end{aligned} \tag{16}$$

as desired.

Next, let's figure out S^2_{total}. Following the lead of Eq. (2), we can write S^2_{total} as

$$S^2_{total} = (S_{x,total})^2 + (S_{y,total})^2 + (S_{z,total})^2. \tag{17}$$

Let's look at the last piece of the sum first:

$$\begin{aligned} (S_{z,total})^2 &= (S_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z)^2 \\ &= (S_z \otimes \mathbb{1})^2 + (\mathbb{1} \otimes S_z)^2 + (S_z \otimes \mathbb{1})(\mathbb{1} \otimes S_z) + (\mathbb{1} \otimes S_z)(S_z \otimes \mathbb{1}) \\ &= (S_z^2 \otimes \mathbb{1}) + (\mathbb{1} \otimes S_z^2) + 2(S_z \otimes S_z). \end{aligned} \tag{18}$$

The other terms are the same, with x and y substituted for z. Therefore,[††]

$$S^2_{total} = (S^2 \otimes \mathbb{1}) + (\mathbb{1} \otimes S^2) + 2(S_x \otimes S_x + S_y \otimes S_y + S_z \otimes S_z). \tag{19}$$

By construction, our $|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle$ states are eigenstates of $S^{2\,(1)}$ $(S^2 \otimes \mathbb{1})$, $S^{2\,(2)}$ $(\mathbb{1} \otimes S^2)$, $S_z^{(1)}$ $(S_z \otimes \mathbb{1})$, $S_z^{(2)}$ $(\mathbb{1} \otimes S_z)$, and $S_{z,total}$. Are they also eigenstates of S^2_{total}? We can find out by evaluating the commutator of S^2_{total} and $S_z^{(1)}$. Using the fact that

$$(A + B) \otimes C = (A \otimes C) + (B \otimes C), \tag{20}$$

(which is easy to prove using the definition of the tensor product in terms of matrices given in section 2.2), we can evaluate the commutator (skipping many steps, which you should try to fill in):

$$\begin{aligned} [S^2_{total}, S_z^{(1)}] &= 2\left([S_x, S_z] \otimes S_x + [S_y, S_z] \otimes S_y\right) \\ &= 2i\hbar\left((S_x \otimes S_y) - (S_y \otimes S_x)\right) \\ &\neq 0. \end{aligned} \tag{21}$$

[††]Note that this is the "correct" way to write Griffith's equation 4.179. Comments similar to those in the previous footnote apply here, with the added point that taking the dot product of operators from different Hilbert spaces makes even less sense (unless you know what that notation *really means*) than adding them.

We see that the commutator is nonzero, which means that we *cannot* construct simultaneous eigenstates of S^2_{total} and $S_z^{(1)}$, and therefore the $|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle$ states are *not* eigenstates of S^2_{total}. We conclude that, $S^{2\,(1)}$, $S^{2\,(2)}$, $S_z^{(1)}$, $S_z^{(2)}$, and $S_{z,total}$ constitute a *complete set of commuting observables* for this system‡‡.

2.2 Tensor Products as Matrices

From early in our study of quantum mechanics, we have emphasized the fact that quantum mechanics is "just linear algebra", in other words, that our operators can be represented by square matrices and our states by column vectors. Is it possible to do this with operators and states built as tensor products of different Hilbert spaces? It is, although it requires us to define a new sort of matrix multiplication.

Suppose we have two 2x2 matrices, **A** and **B**, given by

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \tag{22}$$

Then, the tensor product of **A** and **B**, also known as the Kronecker product in the context of linear algebra, is given by

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \begin{pmatrix} A_{11} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{12} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ A_{21} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{22} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}. \end{aligned} \tag{23}$$

This result generalizes to matrices of different sizes in the expected way.

Let's apply this to our operators and states. To be specific, let's choose the case

‡‡A *complete set of commuting observables* (CSCO) is a set of commuting operators whose eigenvalues completely specify the state of a system. For example, a CSCO for an electron in the hydrogen atom, ignoring spin, is H, L^2, and L_z.

of two spin-$\frac{1}{2}$ particles. First, let's construct the states. For example,

$$\left|\frac{1}{2}\;\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\;\frac{1}{2}\right\rangle = \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \tag{24}$$

whereas

$$\left|\frac{1}{2}\;-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\;\frac{1}{2}\right\rangle = \begin{pmatrix} 0\begin{pmatrix}1\\0\end{pmatrix} \\ 1\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \tag{25}$$

and likewise

$$\left|\frac{1}{2}\;\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\;-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \left|\frac{1}{2}\;-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\;-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{26}$$

Evidently, our new Hilbert space is 4-dimensional, and the states we just calculated are a good bases for that space.

We can play a similar game with the operators. For example,

$$S_z \otimes \mathbb{1} = \begin{pmatrix} \frac{\hbar}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & -\frac{\hbar}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}, \tag{27}$$

and

$$\mathbb{1} \otimes S_z = \begin{pmatrix} 1\begin{pmatrix}\frac{\hbar}{2}&0\\0&-\frac{\hbar}{2}\end{pmatrix} & 0\begin{pmatrix}\frac{\hbar}{2}&0\\0&-\frac{\hbar}{2}\end{pmatrix} \\ 0\begin{pmatrix}\frac{\hbar}{2}&0\\0&-\frac{\hbar}{2}\end{pmatrix} & 1\begin{pmatrix}\frac{\hbar}{2}&0\\0&-\frac{\hbar}{2}\end{pmatrix} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}. \tag{28}$$

Given these two operators, we can also construct the $S_{z,total}$ operator (Eq. (15)):

$$S_{z,total} = S_z \otimes \mathbb{1} + \mathbb{1} \otimes S_z = \hbar\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix}. \tag{29}$$

Practice acting with these operators on these states, both in tensor product notation and in matrix notation, to see that both notations agree with each other.

3 The "Composite" Representation

Now we would like to start thinking about how we could describe our system of two particles, one of spin s_1 and the other of spin s_2, in a way that emphasizes the composite system rather than the individual particles✠. Remember that a particle of spin s has $2s{+}1$ possible states, one for each value of m_s, so that there are $2s_1{+}1$ times $2s_2{+}1$ possible states for our two-particle system. Suppose that we calculate the total spin in the z-direction, that is, the sum of m_{s_1} and m_{s_2}, for each of these states, and carefully tabulate them. Assuming that s_2 is greater than or equal to s_1, we might come up with something like this:

$$\begin{array}{llll}
s_1 + s_2 & (s_1-1)+s_2 & \cdots & (s_1-2s_1)+s_2 \\
s_1+(s_2-1) & (s_1-1)+(s_2-1) & \cdots & (s_1-2s_1)+(s_2-1) \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & (s_1-2s_1)+(s_2-2(s_2-s_1)) \\
\vdots & \vdots & \vdots & \vdots \\
s_1+(s_2-(2s_2-1)) & (s_1-1)+(s_2-(2s_2-1)) & \cdots & (s_1-2s_1)+(s_2-(2s_2-1)) \\
s_1+(s_2-2s_2) & (s_1-1)+(s_2-2s_2) & \cdots & (s_1-2s_1)+(s_2-2s_2)
\end{array}$$

If we rewrite this to get rid of all the silly parentheses and color-code the entries suggestively, we produce:

$$\begin{array}{llll}
\color{red}{s_1+s_2} & \color{blue}{(s_1+s_2)-1} & \cdots & \color{green}{s_2-s_1} \\
\color{red}{(s_1+s_2)-1} & \color{blue}{(s_1+s_2)-2} & \cdots & \color{green}{(s_2-s_1)-1} \\
\color{red}{\vdots} & \color{blue}{\vdots} & \vdots & \color{green}{\vdots} \\
\color{red}{\vdots} & \color{blue}{\vdots} & \vdots & \color{green}{s_1-s_2} \\
\color{red}{\vdots} & \color{blue}{\vdots} & \vdots & \vdots \\
\color{red}{(s_1-s_2)+1} & \color{blue}{s_1-s_2} & \cdots & \color{blue}{-(s_1+s_2)+1} \\
\color{red}{s_1-s_2} & \color{red}{(s_1-s_2)-1} & \cdots & \color{red}{-(s_1+s_2)}
\end{array}$$

✠In principle, you should be able forget everything you read in Section 2 and still understand this one, because our two representations, while closely related, are also logically self-contained. We'll look at the connection between them in Section 4.

Remember, each entry in this table represents the total spin in the z-direction for some state of our two particle spin system. However, suppose we were just handed this table and told that the numbers in it represent the z-component of the spin of some physical system, and suppose further that we are *not told* that the system is two particles whose spins we have summed. If we looked just at the red entries, we would note that they take on all values, in integer steps, between s_1+s_2 and -(s_1+s_2). Likewise, the blue entries take on all values, in integer steps, between s_1+s_2-1 and -(s_1+s_2-1), and so on, until we get to the green entries, which take on all values, in integer steps, between s_2-s_1 and -(s_2-s_1). Hence, we would conclude that this table is a table of all the possible spin states associated with s={$s_1 + s_2$, $s_1 + s_2 - 1$, . . ., $|s_2 - s_1|$}

Since we *do* in fact know that we constructed this table by summing the z-components of the spin of two particles, our conclusion is that the Hilbert space of our two particles of spins s_1 and s_2 can be constructed out of all of the states represented by s={$s_1 + s_2$, $s_1 + s_2 - 1$, . . ., $|s_2 - s_1|$}. Symbolically, we say that the Hilbert space of our two particles is given by

$$|s_1 + s_2 \; m_{s_1+s_2}\rangle \oplus |s_1 + s_2 - 1 \; m_{s_1+s_2-1}\rangle \oplus \cdots \oplus \big||s_2 - s_1| \; m_{|s_2-s_1|}\big\rangle \,, \qquad (30)$$

where the $\oplus$ operation is called a *direct sum*. This notation means that the Hilbert space for the two-particle system has as its basis states all of the states listed in the sum.

Let's look at an explicit example. Suppose we had as our system a spin-1 particle and a spin-2 particle. We expect there to be (2*1+1)(2*2+1), or 15, states, with total spins running between 2+1 and 2-1, inclusive, in integer steps. We can verify this by tabulating these states by their total spin in the z-direction:

$$\begin{array}{ccc} 1+2 & 0+2 & -1+2 \\ 1+1 & 0+1 & -1+1 \\ 1+0 & 0+0 & -1+0 \\ 1-1 & 0-1 & -1-1 \\ 1-2 & 0-2 & -1-2 \end{array} \qquad \text{or} \qquad \begin{array}{ccc} \color{red}{3} & \color{blue}{2} & \color{green}{1} \\ \color{red}{2} & \color{blue}{1} & \color{green}{0} \\ \color{red}{1} & \color{blue}{0} & \color{green}{-1} \\ \color{red}{0} & \color{blue}{-1} & \color{blue}{-2} \\ \color{red}{-1} & \color{red}{-2} & \color{red}{-3} \end{array}$$

We see that the result is just what we wanted: 15 states made up of a spin-3 multiplet, a spin-2 multiplet, and a spin-1 multiplet♣. The Hilbert space for the system is

♣A *multiplet* is simply a set of spin states with the same value of s. We say that we have a spin-s multiplet, instead of a spin-s particle, in cases when we are talking about the spin of a multi-particle system.

therefore

$$|3\ m_{s=3}\rangle \oplus |2\ m_{s=2}\rangle \oplus |1\ m_{s=1}\rangle\,, \tag{31}$$

so the basis states are $|3\ 3\rangle$, $|3\ 2\rangle$, $|3\ 1\rangle$, $|3\ 0\rangle$, $|3\ -1\rangle$, $|3\ -2\rangle$, $|3\ -3\rangle$, $|2\ 2\rangle$, $|2\ 1\rangle$, $|2\ 0\rangle$, $|2\ -1\rangle$, $|2\ -2\rangle$, $|1\ 1\rangle$, $|1\ 0\rangle$, $|1\ -1\rangle$.

Now that we know what's going on with our states, we'd like to know which operators we can use to describe them. By construction, each of our basis states is an eigenstate of $S_{z,total}$ and S^2_{total} (in other words, the spin quantum numbers sitting in our basis kets are related to the *total* spin of the composite system, not to the spin of either particle separately). We'd like to know if our basis states are also eigenstates of the spin operators that act on the individual particles, namely, $S^{2\,(1)}$, $S^{2\,(2)}$, $S_z^{(1)}$, and $S_z^{(2)}$. Since all of composite states are possible states for the same two particles of spin s_1 and s_2, all of these states are eigenstates of $S^{2\,(1)}$ and $S^{2\,(2)}$, with eigenvalues $s_1(s_1+1)\hbar^2$ and $s_2(s_2+1)\hbar^2$, respectively. However, these states are *not* in general eigenstates of $S_z^{(1)}$ or $S_z^{(2)}$[◊] Therefore, we have a complete set of commuting observables for our composite Hilbert space consisting of $S_{z,total}$, S^2_{total}, $S^{2\,(1)}$, and $S^{2\,(2)}$.

3.1 Direct Sums as Matrices

As with tensor products, it is possible to represent the direct sum of two matrices as a new matrix, but it requires us to define a new sort of matrix addition.

Suppose we have two matrices, $\mathbf{A}$ and $\mathbf{B}$, of sizes $m \times n$ and $p \times q$, respectively. The direct sum of these matrices is given by

$$\mathbf{A} \oplus \mathbf{B} = \begin{pmatrix} \mathbf{A}_{m\times n} & \mathbf{0}_{m\times q} \\ \mathbf{0}_{p\times n} & \mathbf{B}_{p\times q} \end{pmatrix}, \tag{32}$$

where $\mathbf{0}$ is a matrix of zeroes.

As with tensor products, let's see how this notation works with an explicit example. To be specific, we will pick the case of two spin-$\frac{1}{2}$ particles, whose composite

[◊]There is one major problem with the tables of total spin in this section. For example, in the spin-2/spin-1 example above, we see that there are two states with $m_{s,total}$=2. The way I color-coded the table suggests that one of these corresponds to $|3\ 2\rangle$ and the other to $|2\ 2\rangle$. If this were true, then these states *would* be eigenstates of $S_z^{(1)}$ and $S_z^{(2)}$. However, the truth is that $|3\ 2\rangle$ and $|2\ 2\rangle$ are *linear combinations* of the states presented in the table. We will defer our discussion of the details until Section 4.

Hilbert space is described by

$$|1\ m_{s=1}\rangle \oplus |0\ 0\rangle\,. \tag{33}$$

By inspection, the basis states for our Hilbert space are $|1\ 1\rangle$, $|1\ 0\rangle$, $|1\ -1\rangle$, and $|0\ 0\rangle$♠. Since we have four states, we should represent them by four-entry state vectors, like this:

$$|1\ 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |1\ 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |1\ -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |0\ 0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{34}$$

From the discussion above, we know that these states should be eigenstates of $S_{z,total}$ and S^2_{total}. Using direct sum notation, and preserving the order we adopted in listing states, we can write

$$\begin{aligned} S_{z,total} &= S_{z,s=1} \oplus S_{z,s=0} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \oplus \hbar \begin{pmatrix} 0 \end{pmatrix} \\ &= \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{35}$$

and

$$\begin{aligned} S^2_{total} &= S^2_{s=1} \oplus S^2_{s=0} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \hbar^2 \begin{pmatrix} 0 \end{pmatrix} \\ &= 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{36}$$

♠Let's adopt a convention by which we always list our states in descending order by s, and, within each s multiplet, in descending order by m_s.

Likewise, we find that

$$S^{2\,(1)} = S^{2\,(2)} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{37}$$

You should check that these matrices actually do what they are supposed to do to our states.

4 The Addition of Angular Momentum

We have now studied two different ways to represent the spin states of a system consisting of a spin-s_1 particle and a spin-s_2 particle. In the separate representation, the basis states for our system are eigenstates of $S^{2\,(1)}$, $S^{2\,(2)}$, $S_z^{(1)}$, $S_z^{(2)}$, and $S_{z,total}$ and live in a Hilbert space given by

$$|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle\,. \tag{38}$$

In the composite representation, the basis states for our system are eigenstates of $S_{z,total}$, S^2_{total}, $S^{2\,(1)}$, and $S^{2\,(2)}$ and live in a Hilbert space given by

$$|s_1+s_2\ m_{s_1+s_2}\rangle \oplus |s_1+s_2-1\ m_{s_1+s_2-1}\rangle \oplus \cdots \oplus \big||s_2-s_1|\ m_{|s_2-s_1|}\big\rangle\,, \tag{39}$$

Since these two Hilbert spaces describe the same physical system, it must be that they are *the same Hilbert space*; in other words, it must be that

$$|s_1\ m_{s_1}\rangle \otimes |s_2\ m_{s_2}\rangle = |s_1+s_2\ m_{s_1+s_2}\rangle \oplus |s_1+s_2-1\ m_{s_1+s_2-1}\rangle \oplus \cdots \oplus \big||s_2-s_1|\ m_{|s_2-s_1|}\big\rangle\,. \tag{40}$$

This is the essence of the addition of angular momentum (both orbital and spin) in quantum mechanics: given two particles with angular momentum$^\heartsuit$, we can equally-well represent the system in terms of eigenstates of the individual particle's angular momentum operators or in terms of the system's total angular momentum operators$^\flat$.

$^\heartsuit$This also works with one particle with *two contributions* to its angular momentum, for example, an electron in a hydrogen atom that has both orbital angular momentum and spin angular momentum.

$^\flat$This is an exceedingly peculiar, nonintuitive result, so don't feel bad if you don't have a gut sense of what that means. What is important at this stage is that you understand the logic behind that result, and that you can use the formalism to solve problems. More intuitive understanding will come with time

Before we go on to an explicit example, there is one more note about notation that is worth making. Equation (40) is rather cumbersome to write, and most of what's in it is really redundant. A much briefer way of writing it is

$$s_1 \otimes s_2 = s_1 + s_2 \oplus s_1 + s_2 - 1 \oplus \cdots \oplus |s_2 - s_1|\,; \tag{41}$$

in other words, we use the total spin quantum number for each ket to represent the entire ket. For example, we can write

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0, \tag{42}$$

or

$$\frac{3}{2} \otimes 2 = \frac{7}{2} \oplus \frac{5}{2} \oplus \frac{3}{2} \oplus \frac{1}{2}. \tag{43}$$

One cute thing to note about this notation is that if one replaces the quantum number representing a state with the multiplicity of the state and removes the circles from around the addition and multiplication symbols, one should always get a correct equation. For instance, spin-$\frac{1}{2}$ has two states, spin-1 has three, and spin-0 has one, so Eq. (42) becomes

$$2 \times 2 = 1 + 0. \tag{44}$$

Likewise, Eq. (43) becomes

$$4 \times 5 = 8 + 6 + 4 + 2. \tag{45}$$

This works because what we are really doing here is counting up the total number of basis states in both representations of our Hilbert space, and the total must be the same in either case. It is possible to show this in generality, starting from Eq. (41). See if you can figure out how to do it.

4.1 Matching the Basis States

It's finally time to tackle the issue that we've been skirting for the majority of these notes: how to figure out which basis states in the separate representation are equal to which basis states in the composite representation. As a specific example, lets have our particles be spin-1 and spin-$\frac{1}{2}$[♮]. Then, in the separate representation, our basis states are

$$\begin{array}{lll} |1\ 1\rangle \otimes \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle & |1\ 0\rangle \otimes \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle & |1\ -1\rangle \otimes \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle \\ |1\ 1\rangle \otimes \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle & |1\ 0\rangle \otimes \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle & |1\ -1\rangle \otimes \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \end{array}. \tag{46}$$

[♮]I choose *not* to do spin-$\frac{1}{2}$/spin-$\frac{1}{2}$ because that example is already done in your book.

Since

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}, \tag{47}$$

our basis states in the composite representation are

$$\begin{array}{cccc} \left|\frac{3}{2}\ \frac{3}{2}\right\rangle & \left|\frac{3}{2}\ \frac{1}{2}\right\rangle & \left|\frac{3}{2}\ -\frac{1}{2}\right\rangle & \left|\frac{3}{2}\ -\frac{3}{2}\right\rangle \\ & \left|\frac{1}{2}\ \frac{1}{2}\right\rangle & \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle & \end{array}. \tag{48}$$

So how do we figure out which equals which? One clue is that basis states in both representations are eigenstates of $S_{z,total}$. For our example here, there is exactly one state in each representation with $S_{z,total} = \frac{3}{2}\hbar$, so they must be equal:

$$\left|\frac{3}{2}\ \frac{3}{2}\right\rangle = |1\ 1\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle. \tag{49}$$

The same holds true for $S_{z,total} = -\frac{3}{2}\hbar$:

$$\left|\frac{3}{2}\ -\frac{3}{2}\right\rangle = |1\ -1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle. \tag{50}$$

At this point, we run into problems. For instance, there are *two* states in each representation that have $S_{z,total} = \frac{1}{2}\hbar$: $\left|\frac{3}{2}\ \frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$, and $|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ and $|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$. In general, the two from one representation will be a linear combination of the two from the other. We need to figure out which linear combination is correct.

The first step is to act on Eq. (49) with the lowering operator

$$S_- = S_- \otimes \mathbb{1} + \mathbb{1} \otimes S_-, \tag{51}$$

remembering that S_- acts like this:

$$S_- |s\ m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s-1)}\,|s\ m_s - 1\rangle\,. \tag{52}$$

We find that

$$\begin{aligned} S_-\left|\frac{3}{2}\ \frac{3}{2}\right\rangle &= (S_- \otimes \mathbb{1} + \mathbb{1} \otimes S_-)\,|1\ 1\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\ S_-\left|\frac{3}{2}\ \frac{1}{2}\right\rangle &= S_-\,|1\ 1\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle + |1\ 1\rangle \otimes S_-\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \\ \hbar\sqrt{3}\left|\frac{3}{2}\ \frac{1}{2}\right\rangle &= \hbar\sqrt{2}\,|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \hbar\,|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle \\ \left|\frac{3}{2}\ \frac{1}{2}\right\rangle &= \sqrt{\frac{2}{3}}\,|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\,|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle. \end{aligned} \tag{53}$$

Now we know what linear combination of $|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ and $|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$ is equal to $\left|\frac{3}{2}\ \frac{1}{2}\right\rangle$.

If we repeat this process on $\left|\frac{3}{2}\ \frac{1}{2}\right\rangle$, we get

$$\left|\frac{3}{2}\ -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}|1\ 0\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}|1\ -1\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle, \tag{54}$$

and if we do it once more, we get $\left|\frac{3}{2}\ -\frac{3}{2}\right\rangle = |1\ -1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$, as expected. Try to fill in the steps of these derivations.

All this still leaves us with no knowledge of the states $\left|\frac{1}{2}\ \pm\frac{1}{2}\right\rangle$. However, we *do* know that $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ must be a linear combination of $|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ and $|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$, and we also know that it must be orthogonal to $\left|\frac{3}{2}\ \frac{1}{2}\right\rangle$, since basis states are always orthogonal. If we write

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = A\,|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle + B\,|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle, \tag{55}$$

and use the fact that

$$\left\langle\frac{3}{2}\ \frac{1}{2}\middle|\frac{1}{2}\ \frac{1}{2}\right\rangle = 0, \tag{56}$$

we find that

$$A\sqrt{\frac{2}{3}} + B\sqrt{\frac{1}{3}} = 0. \tag{57}$$

Combining that with the fact that $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ must be normalized, that is,

$$|A|^2 + |B|^2 = 1, \tag{58}$$

we find that A=$-\sqrt{\frac{1}{3}}$ and B=$\sqrt{\frac{2}{3}}$, so that

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}\,|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle. \tag{59}$$

Finally, if we use the lowering operator on this state, we get

$$\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}|1\ 0\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}|1\ 0\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle. \tag{60}$$

So, after a bit of tedious work, we have accomplished our task of writing the basis states from the composite representation in terms of those in the separate representation. This method generalizes to combining any two angular momenta.

By doing this, we can also explicitly see why states from the composite basis are *not* eigenstates of $S_z^{(1)}$ and $S_z^{(2)}$. For instance, the state

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}\,|1\ 0\rangle \otimes \left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,|1\ 1\rangle \otimes \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle. \tag{61}$$

is clearly not an eigenstate of $S_z^{(1)}$ or $S_z^{(2)}$ because it is a mixture of two states with different values of m_{s_1} and m_{s_2}. Likewise, if we were to invert all of these results by writing the separate basis states in terms of the composite basis states, we would see that the separate basis states are *not* eigenstates of S^2_{total} because they are mixtures of states with different values of s_{total}.

4.2 Systems of More Than Two Particles

Finally, we would like to talk about how to combine angular momentum for more than two particles. The answer is, very easily with tensor product notation. This is best illustrated by example.

For instance, suppose we wanted to combine three spin-$\frac{1}{2}$ particles. We would get

$$\begin{aligned}
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= (\frac{1}{2} \otimes \frac{1}{2}) \otimes \frac{1}{2} \\
&= (1 \oplus 0) \otimes \frac{1}{2} \\
&= (1 \otimes \frac{1}{2}) \oplus (0 \otimes \frac{1}{2}) \\
&= (\frac{3}{2} \oplus \frac{1}{2}) \oplus \frac{1}{2} \\
&= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}.
\end{aligned} \tag{62}$$

We see that three spin-$\frac{1}{2}$ particles combine to form a spin-$\frac{3}{2}$ multiplet and two distinct spin-$\frac{1}{2}$ multiplets. The difference between these two spin-$\frac{1}{2}$ multiplets is symmetry: one of them is antisymmetric with respect to swapping the first two spin-$\frac{1}{2}$ particles and the other is symmetric with respect to that swapping[♯].

[♯]Using Griffith's up-and-down-arrow notation, one of the spin-$\frac{1}{2}$ multiplets is given by

$$\left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \tag{63a}$$

$$\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle), \tag{63b}$$

Let's do one last example:

$$\begin{aligned}
1 \otimes 2 \otimes 3 &= (1 \otimes 2) \otimes 3 \\
&= (3 \oplus 2 \oplus 1) \otimes 3 \\
&= (3 \otimes 3) \oplus (2 \otimes 3) \oplus (1 \otimes 3) \\
&= (6 \oplus 5 \oplus 4 \oplus 3 \oplus 2 \oplus 1 \oplus 0) \oplus (5 \oplus 4 \oplus 3 \oplus 2 \oplus 1) \oplus (4 \oplus 3 \oplus 2) \\
&= 6 \oplus 5 \oplus 5 \oplus 4 \oplus 4 \oplus 4 \oplus 3 \oplus 3 \oplus 3 \oplus 2 \oplus 2 \oplus 2 \oplus 1 \oplus 1 \oplus 0. \qquad (65)
\end{aligned}$$

5 Conclusion

The theory of angular momentum in quantum mechanics is one of the least intuitive parts of the whole subject, but it is also one of the most mathematically beautiful and theoretically important, because it illustrates virtually all of the quantum weirdness we have encountered throughout the course of this class. Hopefully these notes have helped make a little more sense of this subject.

while the other is

$$\left|\frac{1}{2}\;\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}(2\left|\uparrow\uparrow\downarrow\right\rangle - \left|\uparrow\downarrow\uparrow\right\rangle - \left|\downarrow\uparrow\uparrow\right\rangle) \qquad (64a)$$

$$\left|\frac{1}{2}\;-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}(\left|\uparrow\downarrow\downarrow\right\rangle + \left|\downarrow\uparrow\downarrow\right\rangle - 2\left|\downarrow\downarrow\uparrow\right\rangle). \qquad (64b)$$

The first set is antisymmetric with respect to swapping particle 1 and 2, and the second set is symmetric with respect to that swap.