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Introduction to String Theory

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Abstract

This course is aimed at beginning Ph.D. students that are not familiar with elementary particle physics. The basic concepts of string theory are introduced and its main features are exposed. The main ingredients of string theory, as extra dimensions, supersymmetry, dualities and branes are explained and it is shown what is their role in establishing a theory of the particles and all known forces between them. This should lead to an understanding of the status of research in establishing a standard model including gravity and a theory of the early universe.

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Contents

1 Elementary particle physics

What are the goals of elementary particle physics?
Which are the established main facts?
What is a particle?
Which are the main theories?
Why do we need string theory?

Goal of high-energy physics: particles and forces.
Use 'The official string theory web site' for the standard model.
Remark on pentaquarks: see Physicsweb news.

1.1 Particles

The spacetime is a fixed background on which particles move. The Poincaré group plays an important role. An elementary particle is a set of states transforming in an irreducible representation of the Poincaré group: the Lorentz rotations $\mathrm{SO}(1, D-1)$ and translations. An irreducible representation is characterized by Casimirs: mass and spin. The mass is determined by $M^2 = -p^2$.

States with $M^2 < 0$ are called tachyons. These are non-stable states: their kinetic energy decreases by increasing speed. I.e. they signal that we have not taken the right vacuum. We are expanding around a maximal of the potential, rather than a minimum.

For massless particles. Using a Lorentz transformation, we can put the momentum in a light-like direction e.g. $p^\mu = p^+ \delta_+^\mu$. The states form a representation of $\mathrm{SO}(D-2)$. E.g. in 4 dimensions you can have just one state, or, if there is also parity invariance, a massless particle of spin s has two helicity states $h = \pm s$.

For massive particles we can put the momentum in the timelike direction (rest frame). The states form a representation of $\mathrm{SO}(D-1)$ (or rather its covering group, meaning that spinor representations are possible). In 4 dimensions, this is characterized by the spin s. A particle of spin s has $(2s+1)$ states with helicity $h = s,\ s-1, \ldots, -s$.

Remark that we have started here with a 'flat space' such that the Poincaré group is the main tool. If spacetime is curved, e.g. a (anti-)de Sitter space, the notion of a particle needs to be reconsidered, but the same methods are used to define it.

1.2 Quantum field theory and the path integral

Theories to describe the particles and forces: quantum mechanics and special relativity, unified in quantum field theory. The theory is defined by an action, that determines classical field equations. The quadratic terms in the action determine the propagation of the particles and the other terms determine their interactions. Often they are multiplied by a coupling constant, e.g. g, such that for weak coupling $g \ll 1$ one can work perturbatively in this coupling constant. One can represent the interactions as Feynman graphs, where tree graphs determine the classical scattering amplitudes, and loops determine the quantum effects.

A convenient way to describe field theory amplitudes uses the Feynman path integral. In path integral quantization, amplitudes are given by summing over all possible histories interpolating between the initial and final states. Each history is weighted by

$$\exp\left(\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}\right), \qquad S_{\mathrm{cl}} = \int \mathrm{d}x^D \mathcal{L}(\Phi(x), \partial\Phi(x)), \tag{1.1}$$

where S_{cl} is the classical action for the given history. The path integral is the quantity

$$Z = \int \mathcal{D}\Phi \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}}, \tag{1.2}$$

where the integral is taken over all configurations of the fields Φ. Expectation values of functionals of fields $\mathcal{A}(\Phi)$ are determined as

$$\langle \mathcal{A} \rangle = \frac{\int \mathcal{D}\Phi \mathcal{A}(\Phi) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}}}{\int \mathcal{D}\Phi \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}}}. \tag{1.3}$$

One usually first tries to interpret this definition in perturbation theory in Planck's constant $\hbar$. In a certain regime, the amplitudes can have a meaningful, although most likely asymptotic, expansion of the form

$$\langle \mathcal{A} \rangle \sim \sum_n A_g \hbar^g, \tag{1.4}$$

where the perturbative coefficients A_g are computed as sums over all Feynman diagrams Γ with a fixed number of g loops

$$A_g = \sum_\Gamma \frac{w_\Gamma}{\#\,\mathrm{Aut}(\Gamma)}. \tag{1.5}$$

Here the individual weight w_Γ of the diagram Γ is computed using the Feynman rules and we divide by the order of the symmetry group of the diagram. This relation with graphs explains why quantum field theories can describe point-particles and their interactions.

Loops involve an integral over momentum that can circulate. Lines in the loop are not 'on-shell', i.e. $p_\mu p^\mu \neq -m^2$. These integrals may diverge. This is solved in most theories by regularization and renormalization. Regularization is the process of giving meaning to the infinite expressions. A parameter is defined such that the integral only diverges if that parameter obtains a critical value. Regularization means that terms are added to the action that depend on this parameter, and that also diverge if this parameter obtains the critical value. But this is arranged in such a way that the sum of the original and new Feynman graphs give expressions such that one obtains a finite result. Though this seems a strange procedure, the results for most theories are surprisingly accurate in agreement with experiments. However, for quantum gravity this does not work.

Finally, I want to draw the attention to 'anomalies', which are another aspects of quantization of gauge theories. Often the quantization breaks gauge symmetries. Technically, this comes about because there is no regularization scheme that respects the symmetry (the symmetry is only valid when the regularization parameter is equal to its critical value). An anomaly in a rigid symmetry is often a physical feature that can be measured, and can be found in agreement with an experiment. However, an anomaly in a local symmetry means that some degrees of freedom appear that were not present in the classical theory. This might lead to inconsistencies. Often the absence of such anomalies gives a restriction on the field content of consistent theories. It might give an equation on the number of certain matter representations in the gauge group.

Exercise 1.1: (Advanced). What are the anomaly restrictions on the standard model?
How do they come about?

1.3 Elementary interactions and their theories

Electromagnetism is mediated through a vector field A_μ that describes a spin-1 particle. A_μ has D components, while a spin-1 particle is a representation of $\mathrm{SO}(D-2)$ and carries $D-2$ physical degrees of freedom. Two degrees of freedom are eliminated from the 4 components of A_μ by the gauge invariance $\delta A_\mu = \partial_\mu \Lambda$ and the field equation eliminates then one more degree of freedom. One may eliminate these degrees of freedom by explicit 'gauge choices', determining some components of A_μ. Often it is more useful to keep the Lorentz invariance, and describing the theory using 'Faddeev-Popov ghost' fields. These are non-physical fields that 'compensate' in some way the gauge degrees of freedom of other fields in the theory. E.g. in this case, the ghost corresponding to the Λ transformation and a corresponding antighost reduce the effective physical degrees of freedom of the gauge field to $D-2$.

Charged fields, like the electron, are fields that transform under the gauge transformation. For electromagnetism this transformation is a multiplication by a complex phase factor, i.e. $\psi \to \mathrm{e}^{\mathrm{i}eq\Lambda}\psi$ where e is a the unit charge and q is a real number, determining the charge of the particle as qe. For later reference: $\delta\psi = \mathrm{i}eq\Lambda\psi$. They feel the electromagnetism because in a gauge-invariance lagrangian, their kinetic term should be in the form $D_\mu\psi = \partial_\mu\psi - \mathrm{i}eqA_\mu\psi$ in order that this does not transform with derivatives on the parameter. In this case, the gauge transformation was just a $\mathrm{U}(1)$ group.

Weak interactions are described together with electromagnetism (electro-weak unification) through gauge symmetry with an $\mathrm{SU}(2) \times U(1)$ gauge group. This involves 4 generators and in a gauge theory we then need 4 gauge fields. Gauge fields are fields whose transformation involves $\partial_\mu\Lambda^I$ where Λ^I are the parameters. 'Matter fields' are fields that are in a representation of the gauge group. They feel the interaction if this representation is non-trivial, again though the appearance of covariant derivatives in the action. Another important part of the action comes into play here: the scalar potential $V(\phi)$, where ϕ stands for the scalar fields. It is the part of the Lagrangian that depends only on the scalars and does not contain derivatives. Its minimum determines vacuum states where the other fields are zero (preserving Lorentz invariance) and the scalar fields are constants. Such vacua may not be invariant under the symmetry group. In that case we say that the gauge symmetry is spontaneously broken. The gauge fields corresponding to the broken symmetries are massive. This is what happens in the electroweak theory. The vacuum is only invariant under a linear combination of the $U(1)$ factor in the gauge groups and a generator that belongs to SU(2). This linear combination corresponds to the gauge symmetry of electromagnetism. The other 3 gauge fields are the 'W and Z massive gauge bosons'.

Strong interactions are described by a gauge theory of SU(3), that acts on the colours of the quarks. Quarks are in triplet representations, and the gauge fields are the gluons. A main aspect of strong interactions is that they keep the quarks confined within mesons and hadrons. It is not proven that the theory of SU(3) performs this binding. The problem is that this is an effect due to strong coupling. Hence, it can not be proven by perturbation theory. We need a non-perturbative analysis of the theory to obtain such a result. Strong indications have been given by lattice gauge theories. This is an approximation where the continuous spacetime is replaced by a lattice and strong computer methods are used to derive properties of the particles. String theory will indicate a new way, as we will see below.

The previous parts together, a theory with gauge group $\mathrm{SU}(3)\times SU(2)\times U(1)$ is called '*the standard model*'. It thus contains the gauge fields and the 'matter multiplets', representations of the group whose components are the leptons and the quarks.

Exercise 1.2: Which are the full set of representations that are included to build the presently known standard model? What are the modifications due to the recent observations of massive neutrinos?

Extensions have been made 'unifying' $\mathrm{SU}(3)\times SU(2)\times U(1)$ in one using simple groups like SU(5) or E_6. These have more generators, hence more forces. Some of these forces may mediate the decay of the proton. To limit this decay in the observational limits is one of the main problems. This approach is called '*grand unification*'. The success was limited and moreover, it still neglects one main force: gravity.

Exercise 1.3: We mention here for the first time an exceptional group: E_6. Do you know the catalogue of all simple Lie groups (or Lie algebras)? Make a table including their ranks and their dimensions. Consider also the accidental degeneracies for groups of low rank.

The theory of *gravity* is general relativity, which makes use of a curved spacetime. In field theory as mediated by a spin 2 particle $g_{\mu\nu}=\eta_{\mu\nu}+\kappa\tilde{\zeta}_{\mu\nu}$. Concerning units: the Newton constant is defined as $G_{\rm N}$ in

$$F_{12}=\frac{G_{\rm N}m_1m_2}{|r_{12}|^{D-2}}\,, \tag{1.6}$$

where for later convenience, I write $D-2$ rather than 2 to be able to generalize to more dimensions. For $D=4$ we have $G_{\rm N}=6.7\times 10^{-11}m^3kg^{-1}s^{-2}$. The Planck mass $M_{\rm Planck}^{2-D}=G_{\rm N}$ and length $\ell_{\rm Planck}=M_{\rm Planck}^{-1}$ are then defined using the $\hbar=c=1$ conventions, and are in 4 dimensions

$$M_{\rm Planck}=2.2\times 10^{-8}kg=1.22\times 10^{19}GeV\,,\qquad \ell_{\rm Planck}=1.6\times 10^{-35}m \tag{1.7}$$

We use GeV as unit for energy which is $1.6 \times 10^{-10} J$ or, dividing by c^2, is equivalent with a mass of $1.8 \times 10^{-27} kg$, or $5 \times 10^{15} m^{-1}$. We often use also in $D = 4$ $M_P \equiv M_{\rm Planck}/\sqrt{8\pi} \sim 2.4 \times 10^{18}$ GeV and $\kappa = M_P^{-1}$ and in general dimensions $M_P \equiv M_{\rm Planck}(8\pi)^{1/(2-D)}$ and $\kappa \equiv M_P^{(2-D)/2} = \sqrt{8\pi G_N}$.

The action for general relativity (GR) is then

$$S_{\rm GR} = \int \mathrm{d}^D x \frac{1}{2\kappa^2} \sqrt{g} R(g) \,, \tag{1.8}$$

where $R(g)$ is the scalar curvature in the conventions that compact spaces have positive curvature.

Gravity can be seen as the gauge theory of the symmetry of general coordinate transformations. The metric is then the gauge field for these gauge transformations.

Exercise 1.4: (repetition of elements of GR): Which are the building stones to construct an invariant action in GR? E.g. when we use $\varepsilon^{\mu\nu\rho\sigma}$, do we still need a $\sqrt{g}$? Is the value of this object a number ± 1 consistent with general coordinate transformations? Can we raise and lower its indices. How do we construct invariant actions when fermions are present? What is the relation between the spin connection and the Levi-Civita connection? What is the relation between the curvature tensors defined using these two connections? What is the definition of the scalar curvature mentioned above? Check the signs, and verify the last statement for a 2-sphere.

Suppose that I modify the metric with a scale factor $g'_{\mu\nu} = \phi g_{\mu\nu}$. What is then the modification of the action (1.8)?

Other attempts have been made to *unify gravity with the other interactions* in a quantum theory. A noteworthy attempt is supersymmetry and supergravity. While in the previously mentioned gauge theories, bosons and fermions are always in separate representations of the gauge group, here bosons and fermions are necessary to form a representation of the invariance group.

Exercise 1.5: Prove that the action

$$S = \int \mathrm{d}^4 x \left[-\tfrac{1}{2}\partial_\mu A \partial^\mu A - \tfrac{1}{2}\partial_\mu B \partial^\mu B - \tfrac{1}{2}\bar\zeta \partial\!\!\!/ \zeta \right] , \tag{1.9}$$

where ζ is a Majorana ('real') spinor, is invariant under the transformations

$$\begin{aligned} \delta A &= \bar\epsilon\zeta \,, \\ \delta B &= \mathrm{i}\bar\epsilon\gamma_5\zeta \,, \\ \delta\zeta &= \partial\!\!\!/ \left(A + \mathrm{i}\gamma_5 B \right) \epsilon \,, \end{aligned} \tag{1.10}$$

where ϵ is the parameter of the transformation (rigid, i.e. not x-dependent). What you need about Majorana spinors in 4 dimensions is that for two of these

$$\bar\lambda\chi = \bar\chi\lambda \,, \qquad \bar\lambda\gamma_5\chi = \bar\chi\gamma_5\lambda \,, \qquad \bar\lambda\gamma_\mu\chi = -\bar\chi\gamma_\mu\lambda \,. \tag{1.11}$$

where $\bar{\lambda} \equiv \lambda^T \mathcal{C}$ with an antisymmetric 'charge conjugation matrix' $\mathcal{C}$. The algebra of γ-matrices implies that $\partial\!\!\!/ \partial\!\!\!/ = \Box$.

The commutator of two supersymmetry transformations gives (after use of field equations) a translation.

When supersymmetry is promoted to a gauge symmetry, then this involves automatically local translations, i.e. general coordinate transformations. Therefore it involves GR. As mentioned, the gauge field of the general coordinate transformations is the metric, a spin-2 field. The gauge fields of supersymmetry is a spin-3/2 field, which is called the gravitino.

The supersymmetry restricts the quantum corrections produced by loops. This occurs because loops of fermions in field theory have a minus sign with respect to loops of bosons. Supersymmetry arranges it such that many divergences are cancelled. It was for some time hoped that supergravity would be a finite theory of gravity. However, the divergences still appear, although only for a large number of loops (3 loops in the simplest supergravity). There are extensions possible with more supersymmetry generators (more such parameters). However, there are limits. The highest extensions has 8 such spinor parameters ϵ^i, with $i = 1, \ldots, 8$. However, that still does not produce full finite results, and it has not enough freedom to include the standard model.

Exercise 1.6: What is the minimal supergravity action? Can you prove that it is invariant?

So we are left with the question: how can nature be described, taking into account that gravity exists.

Video: chapter 6 and 7 of hour 1: the problems of two types of theories and the idea of strings (12').

Video: part 1 and 2 of 2nd hour (somewhat repetition of previous parts) (13').

2 String solutions and states

How are string vibrations described?
Do, or how do, strings end?
What are the masses of the string states?
Why do strings live in more than 4 spacetime dimensions?

A string along the x direction, vibrating in y direction has the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}\,, \tag{2.1}$$

where v is the wave velocity. Solutions for a string of length L are

$$y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi v t}{L} + b_n \sin \frac{n\pi v t}{L} \right) \sin \frac{n\pi x}{L}\,. \tag{2.2}$$

The frequency of the n-th mode is

$$f_n = \frac{nv}{2L}\,. \tag{2.3}$$

In a relativistic theory there is not such a clear difference between time and space and we have to write covariant equations. The string sweeps out a 2-dimensional surface with (τ, σ) and the equation is

$$\frac{\delta^2 X^\mu(\sigma,\tau)}{\partial \tau^2} = c^2 \frac{\delta^2 X^\mu(\sigma,\tau)}{\partial \sigma^2}\,. \tag{2.4}$$

Solution:

$$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{n\neq 0}^{\pm\infty} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-\mathrm{i}n\pi c\tau/L} \cos \frac{n\pi\sigma}{L}\,. \tag{2.5}$$

with $(\alpha_n)^* = \alpha_{-n}$. This is an open string with ends that are floppy.

2.1 Actions

Can be obtained from requirement of minimal surface. Element of surface is $\sigma^\alpha = (\tau, \sigma)$

$$\mathrm{d}\sigma^{\mu\nu} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \varepsilon^{\alpha\beta} \mathrm{d}\sigma \mathrm{d}\tau \tag{2.6}$$

The Nambu-Goto (NG) action is proportional to the surface

$$S_{\mathrm{NG}} \propto \sqrt{\tfrac{1}{2} |\mathrm{d}\sigma^{\mu\nu} \mathrm{d}\sigma_{\mu\nu}|}\,. \tag{2.7}$$

We find

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \, |\det G_{\alpha\beta}|^{1/2}\,. \tag{2.8}$$

where

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X_\mu}{\partial\sigma^\beta} \,. \tag{2.9}$$

α' has dimension $[\ell]^2$, and defines the tension

$$T_{\rm string} = \frac{1}{2\pi\alpha'} \,. \tag{2.10}$$

The length scale $\sqrt{\alpha'}$ determines at which scale the string oscillations become visible.

Another form is the Polyakov action

$$S_{\rm P} = -\frac{1}{4\pi\alpha'}\int {\rm d}^2\sigma\sqrt{h}h^{\alpha\beta}G_{\alpha\beta} \,. \tag{2.11}$$

See GR. Field equations for $h_{\alpha\beta}$ lead to $h_{\alpha\beta} = \Lambda G_{\alpha\beta}$. Therefore this is equivalent to NG.

Invariance under 2-dimensional GCT and dilations. We can choose 3 conditions:

$$h_{\alpha\beta} = \eta_{\alpha\beta} \,. \tag{2.12}$$

The field equation for the scalars X^μ is then (2.4).

Easy to solve in LC variables

$$\sigma^\pm = \sigma^0 \pm \sigma^1 = \tau \pm \sigma \,. \tag{2.13}$$

The metric

$${\rm d}s^2 = -({\rm d}\sigma^0)^2 + ({\rm d}\sigma^1)^2 = -{\rm d}\sigma^+{\rm d}\sigma^- \,, \qquad \text{or} \qquad \eta_{+-} = \eta_{-+} = -\tfrac{1}{2} \,. \tag{2.14}$$

Field equations for the X^μ are:

$$\eta^{\alpha\beta}\frac{\partial^2 X^\mu}{\partial\sigma^\alpha\partial\sigma^\beta} = 0 \qquad \rightarrow \qquad \partial_-\partial_+ X^\mu = 0 \qquad \rightarrow \qquad X^\mu = X^\mu_{(+)}(\sigma^+) + X^\mu_{(-)}(\sigma^-) \,. \tag{2.15}$$

The field equations for the $h^{\alpha\beta}$ lead to ‘constraints’:

$$G_{++} = G_{--} = 0 \,. \tag{2.16}$$

2.2 Solutions using boundary conditions

We will use σ parameters such that length of string is π.

One needs

$$\left[\frac{\partial X^\mu}{\partial\sigma}\eta_{\mu\nu}\delta X^\nu\right]_{\sigma=0}^{\sigma=\pi} \tag{2.17}$$

There are 3 types of solutions:

$$\begin{aligned} \text{Closed strings} &: \quad X^\mu(\sigma+\pi) = X^\mu(\sigma) \,, \\ \text{Neumann} &: \quad \left.\frac{\partial X^\mu}{\partial\sigma}\right| = 0 \,, \\ \text{Dirichlet} &: \quad \delta X^\mu| = 0 \,. \end{aligned} \tag{2.18}$$

Neumann is what we exposed above, see (2.5), where we have now $L = \pi$ and of course $c = 1$. Dirichlet is for later.

Closed

$$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu \tau + \tfrac{1}{2}\mathrm{i}\sqrt{2\alpha'}\sum_{n\neq 0}^{\pm\infty}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-2\mathrm{i}n\sigma^+} + \tfrac{1}{2}\mathrm{i}\sqrt{2\alpha'}\sum_{n\neq 0}^{\pm\infty}\frac{1}{n}\tilde{\alpha}_n^\mu \mathrm{e}^{-2\mathrm{i}n\sigma^-}\,. \tag{2.19}$$

Here $(\alpha_n)^* = \alpha_{-n}$ and $(\tilde{\alpha}_n)^* = \tilde{\alpha}_{-n}$

See: closed string is like two open strings with modes running in opposite directions: *right movers and left movers.*

Exercise 2.1: For a particle, the action is the length of the worldline (for a massive particle). Determine dimensions: where should the mass occur. Expand for $v \ll c$: see potential energy mc^2 and kinetic energy as in classical mechanics.

Determine momenta, and prove that $p^2 + m^2 = 0$.

When we include

$$\text{Closed strings:}\qquad \alpha_0^\mu \equiv \tilde{\alpha}_0^\mu \equiv \tfrac{1}{2}\sqrt{2\alpha'}p^\mu\,,\qquad \text{Open strings:}\qquad \alpha_0^\mu \equiv \sqrt{2\alpha'}p^\mu\,, \tag{2.20}$$

then the constraints are

$$\begin{aligned} G_{++} &= 4\alpha'\sum_{m=-\infty}^{\infty} L_m \mathrm{e}^{-2\mathrm{i}m\sigma^+} = 0\,, & G_{--} &= 4\alpha'\sum_{m=-\infty}^{\infty}\tilde{L}_m \mathrm{e}^{-2\mathrm{i}m\sigma^-} = 0\,,\\ L_m &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_n^\mu\,\alpha_{m-n}^\mu = 0\,, & \tilde{L}_m &= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_n^\mu\,\tilde{\alpha}_{m-n}^\mu = 0 \end{aligned} \tag{2.21}$$

Note, sloppiness: indices should be $\alpha_n^\mu\alpha_{m-n,\mu}$, but for convenience of writing we keep the μ index up.

2.3 Quantization

Quantum theory: find conjugate momenta: at a particular time: $P_\mu = \frac{\delta\mathcal{L}}{\delta\dot{X}^\mu}$, and impose

$$[X^\mu, P_\nu] = \mathrm{i}\hbar\delta_\nu^\mu\,. \tag{2.22}$$

Exercise 2.2: Calculate the Hamiltonian H and the generator for σ translations, P:

$$\begin{aligned} H &= \int_0^\pi \mathrm{d}\sigma\left(\frac{\partial X^\mu}{\partial\tau}P_\mu - \mathcal{L}\right) = \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\left[(\partial_+X^\mu)^2 + (\partial_-X^\mu)^2\right] = 2(L_0 + \tilde{L}_0)\,,\\ P &= \int_0^\pi \mathrm{d}\sigma\left(\frac{\partial X^\mu}{\partial\sigma}P_\mu\right) = \frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma\left[(\partial_+X^\mu)^2 - (\partial_-X^\mu)^2\right] = 2L_0 - 2\tilde{L}_0\,, \end{aligned} \tag{2.23}$$

where e.g. $(\partial_+X^\mu)^2$ is a shortcut for $\partial_+X^\mu\partial_+X_\mu$.

Here also at equal time:

$$[X^\mu(\sigma), P_\nu(\sigma')] = \mathrm{i}\hbar\delta^\mu_\nu\delta(\sigma-\sigma')\,. \tag{2.24}$$

We find

$$\pi P_\mu = \frac{1}{2\alpha'}\dot{X}_\mu = p_\mu + \frac{1}{\sqrt{2\alpha'}}\sum_{n\neq 0}^{\pm\infty}\left(\alpha^\mu_n \mathrm{e}^{-2\mathrm{i}n\sigma^+} + \tilde{\alpha}^\mu_n \mathrm{e}^{-2\mathrm{i}n\sigma^-}\right)\,. \tag{2.25}$$

This leads to

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\,, \qquad [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\,, \qquad [\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0\,. \tag{2.26}$$

String oscillators determine representations of the Poincaré group that are the particle states.

For particle $|k\rangle$, with $p^\mu|k\rangle = k^\mu|k\rangle$, and constraint $k^2+m^2=0$. Furthermore, use as Fock vacuum

$$\alpha^\mu_m|k\rangle = \tilde{\alpha}^\mu_m|k\rangle = 0 \qquad \text{for } m>0\,. \tag{2.27}$$

Fock space: built from acting with negative modes.

Inner products: reality becomes hermiticity.

$$(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}\,, \qquad \langle k|\alpha^\mu_{-n} = 0 \qquad \text{for } m>0\,. \tag{2.28}$$

Constraint: sufficient that $\langle\text{phys}|L_m|\text{phys}\rangle = 0$. It is inconsistent to put $L_m|\text{phys}\rangle = 0$ for all m. Sufficient for $m \geq 0$.

Problem of normal ordering

$$L_0 \equiv \tfrac{1}{2}\alpha^\mu_0\alpha^\mu_0 + \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha^\mu_n \tag{2.29}$$

But then: we have to be careful. Constraint $L_0|\text{phys}\rangle = a|\text{phys}\rangle$.

Also

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n}\,. \tag{2.30}$$

Use Jacobi

$$[L_1,[L_n,L_{-n-1}]] + [L_n,[L_{-n-1},L_1]] + [L_{-n-1},[L_1,L_n]] = 0\,, \tag{2.31}$$

to prove

$$A(1)(2n+1) + A(n)(-n-2) - A(n+1)(1-n) = 0\,, \tag{2.32}$$

determining all $A(n)$ for $n>2$, once we know $A(1)$ and $A(2)$. Calculate

$$\langle k|\,[L_1,L_{-1}]\,|k\rangle = \langle k|L_1L_{-1}|k\rangle = \langle k|\alpha^\mu_1\alpha^\mu_0\alpha^\nu_{-1}\alpha^\nu_0|k\rangle = 2\langle k|L_0|k\rangle\,. \tag{2.33}$$

Thus $A(1)=0$. Similarly $A(2)=D/2$. Leads to

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n}\,. \tag{2.34}$$

Exercise 2.3: Advanced continuous exercise. The advanced students should write all the relations in the language of conformal field theory with operator product expansions. The commutation relations lead to an expression on $X(z)X(w)$, and the Virasoro algebra (2.34) translates to an expression for $T(z)T(w)$, always as an expansion in $(z-w)$. This gives a useful formalism to work with strings. Also the ghosts and antighost fields have such expansions, and expressions for $T(z)$ which lead to expressions of the central charge, corresponding to the last term in (2.34). The advanced students should rephrase all the developments mentioned below it in the language of conformal field theory.

The algebra clearly implies that we cannot impose $L_m|\mathrm{phys}\rangle = 0$. Rather

$$\begin{aligned} L_m|\mathrm{phys}\rangle &= 0 \qquad \text{for } m>0\,, \qquad L_0|\mathrm{phys}\rangle = a|\mathrm{phys}\rangle\,, \\ \tilde{L}_m|\mathrm{phys}\rangle &= 0 \qquad \text{for } m>0\,, \qquad \tilde{L}_0|\mathrm{phys}\rangle = \tilde{a}|\mathrm{phys}\rangle\,. \end{aligned} \tag{2.35}$$

The requirement that the norm of all states should be positive leads to constraints:

$$D = 26\,, \qquad a = 1\,. \tag{2.36}$$

We will not give a full proof here, but will show how it comes about.

Video: part 7 of 2nd hour (the idea of more dimensions). (7')

2.4 Physical string modes

2.4.1 Open string

First consider the open string. L_0 has a meaning as mass-determining quantity. We write

$$L_0 = -\alpha' M^2 + N\,, \qquad N \equiv \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n\,. \tag{2.37}$$

We find at $N=0$ only $\alpha' M^2 = -a$. All other constraints are trivial, and we define the norm $\langle k|k\rangle = 1$ as normalization of the states.

For $N=1$, a general state can be written as $|a,k,1\rangle \equiv a_\mu \alpha_{-1}^\mu |k\rangle$. Constraints:

$$\begin{aligned} L_0 &: \ \alpha' M^2 = 1-a\,, \\ L_1 &: \ k_\mu a^\mu = 0\,, \\ \text{norm} &: \ \langle k|a_\nu^* \alpha_1^\nu a_\mu \alpha_{-1}^\mu |k\rangle = a_\mu^* a^\mu \geq 0\,. \end{aligned} \tag{2.38}$$

$a>1$ excluded: would be negative mass, hence a^0 mode not excluded and has negative norm.

$a=1$: then k in light-like direction, e.g. k^+, imposing $a_+ = 0$ and zero norm of state a_-: is minimal content.

At $N=2$ we can have

$$|a,b,k,2\rangle \equiv a_\mu \alpha_{-2}^\mu + b_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu |k\rangle\,. \tag{2.39}$$

$b_{\mu\nu}$ is symmetric. Constraints: (with $K^\mu = \sqrt{2\alpha'}k^\mu$, eigenvalue of α_0^μ)

$$\begin{aligned}
L_0 &: \quad \alpha' M^2 = 2 - a\,, \\
L_1 &: \quad 2\left(a_\mu + b_{\mu\nu}K^\nu\right)\alpha_{-1}^\nu|k\rangle = 0\,, \qquad \rightarrow \qquad a_\mu = -b_{\mu\nu}K^\nu\,, \\
L_2 &: \quad 2a_\mu K^\mu + b_\mu{}^\mu = 0\,, \qquad \rightarrow \qquad b_\mu{}^\mu = 2b_{\mu\nu}K^\mu K^\nu\,, \\
\text{norm} &\propto \quad a_\mu^* a^\mu + b_{\mu\nu}^* b^{\mu\nu} = K^\rho K^\nu b_{\mu\rho}^* b^\mu{}_\nu + b_{\mu\nu}^* b^{\mu\nu}
\end{aligned} \tag{2.40}$$

As $a < 2$, states have positive mass, and we can put $K^0 = \sqrt{2(2-a)}$ and $K^i = 0$. Then we have

$$b_{ii} = b_{00}\left(1 + 4(2-a)\right)\,. \tag{2.41}$$

The norm can be written as

$$2(2-a)\left(-b_{00}^* b_{00} + b_{0i}^* b_{0i}\right) + b_{00}^* b_{00} - 2b_{0i}^* b_{0i} + b_{ij}^* b_{ij}\,. \tag{2.42}$$

The b_{0i} states impose again $a \le 1$ and are spurious if $a = 1$.

The b_{ij} with $i \neq j$ contribute positively: positive norm states.

Remains the diagonal components, say $b_{ii} = b_i$. Appear as

$$-x\left(\sum_i b_i^*\right)\left(\sum_j b_j\right) + \sum_i b_i^* b_i\,, \qquad x \equiv \frac{2(2-a)-1}{\left(1+4(2-a)\right)^2} \tag{2.43}$$

Obtains minimum when all b_i are equal: $b_i = x\sum_j b_j$. There are $(D-1)$ of them. Or we can write this as

$$\sum_i\left(b_i^* - x\sum_j b_j^*\right)\left(b_i - x\sum_k b_k\right) + \left(x - x^2(D-1)\right)\left(\sum_i b_i^*\right)\left(\sum_j b_j\right)\,. \tag{2.44}$$

This is positive definite if $D - 1 \le 1/x$. With $a = 1$ this is $D \le 26$. Further analysis brings us to the limit (taking away the trace mode of the matrix b). Hence we find at this level a traceless symmetric tensor, i.e. $D(D-1)/2 - 1$ states forming an irreducible representation of SO$(D-1)$: a massive particle.

Observe tachyon state at $N = 0$: should be taken care off below. This (bosonic) string model is not a stable situation.

Number of states at level n, denoted d_n is the coefficient of w^n in

$$\begin{aligned}
\sum_{n=0}^\infty d_n w^n = \mathrm{Tr}\, w^N &= \prod_{n=1}^\infty (1-w^n)^{-24} = (1 + 24w + \frac{24.25}{2}w^2 + \ldots)(1 + 24w^2 + \ldots)\ldots \\
&= 1 + 24w + \frac{24.27}{2}w^2 + \ldots
\end{aligned} \tag{2.45}$$

2.4.2 Closed strings

For closed strings, we have to use α and $\tilde{\alpha}$ operators. It is nearly as doubling everything. However, one has to keep in mind that α_0^μ is the same as $\tilde{\alpha}_\mu^0$. Therefore e.g. L_0 and $\tilde{L}_0$ are not independent. From the symmetry we can expect $a = \tilde{a}$ in (2.35), and these constraints thus immediately imply

$$(L_0 - \tilde{L}_0)|\text{phys}\rangle = 0 \ \to \ \tfrac{1}{2}\alpha_0^\mu \alpha_0^\mu + N = \tfrac{1}{2}\tilde{\alpha}_0^\mu \tilde{\alpha}_0^\mu + \tilde{N} \ \to \ N = \tilde{N} \,. \tag{2.46}$$

This is called the *level-matching condition*. Thus: after the vacuum at $N = \tilde{N} = 0$ (tachyon state), the first states are formed as

$$\zeta_{\mu\nu}\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |k\rangle \,. \tag{2.47}$$

Again the analysis shows that these states should be massless (as follows form $a = \tilde{a} = 1$), and that they are restricted to $k^\mu \zeta_{\mu\nu} = k^\nu \zeta_{\mu\nu} = 0$. Thus, using light-cone coordinates, $k^\mu = k^+ \delta_+^\mu$, we have $\zeta_{+\mu} = 0$ and states with $\zeta_{-\mu}$ are spurious (norm zero). Hence we have the following representations of $\mathrm{SO}(D-2)$:

symmetric traceless: this is like a graviton, and leads to the identification of this theory with a gravity theory.

the trace: a scalar that will be called the dilaton.

antisymmetric: sometimes called Kalb-Ramond field.

The further analysis is similar to the open string. We find states with higher mass, ... But we have still a tachyon.

We have seen that a consistent quantization of the bosonic string requires 26 spacetime dimensions. This dimension is called the critical dimension. String theories can also be defined in less then 26 dimensions and are therefore called non-critical. They are not Lorentz-invariant.

Here we gave some indications of the necessity of this requirement. There are more complete methods to prove this, e.g. light-cone quantization, or covariant quantization. E.g. a method that is similar to the one in use for covariant quantization in gauge theories makes use of 'Faddeev-Popov ghosts'. These are non-physical fields that 'compensate' in some way the gauge degrees of freedom of other fields in the theory. E.g. a massless spin-1 field is described by a vector field A_μ that has D degrees of freedom. As we have seen in section 1, the physical degrees of freedom are only $D-2$. This is accomplished by adding a ghost field and an antighost.

In this case, these ghost fields will give new modifications to the Virasoro generators L_m. These give new contributions to the 'central charge' c in[1]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \,. \tag{2.48}$$

[1]Note that with the consistency requirement (2.32) and a suitable definition of L_0 we can always bring the last term of (2.30) in this form.

The last term is an 'anomaly' for the reparametrization invariances in the action (2.11).

We saw in (2.34) that each of the components of X^μ contributes a term $+1$ to c. The ghost and antighost for the reparametrizations each contribute with -13 to this number, such that the full quantized energy-momentum has no anomaly (we clarify this further at the end of section 3.1).

2.5 Orientable or non-orientable strings

Are the strings orientable (you can tell which direction you're traveling along the string) or unorientable (you can't tell which direction you're traveling along the string)?

Is there a difference between $\sigma = 0$ and $\sigma = \pi$. If not, "non-oriented strings", then we impose a further restriction on the states. These are obtained from the theories constructed so far by a projection. The "world-sheet parity operation" Ω, an involution, i.e. $\Omega^2 = 1$, is defined as a reflection on the world-sheet:

$$\Omega : \sigma^1 \longrightarrow \pi - \sigma^1 = -\sigma^1 \ \text{ modulo } \ \pi \ . \tag{2.49}$$

The operation implies from the expansions (2.5) and (2.19)

$$\begin{aligned} \alpha_n^\mu &\to (-)^n \alpha_n^\mu \qquad \text{for open strings} \\ \alpha_n^\mu &\to \tilde{\alpha}_n^\mu \qquad \text{for closed strings} \end{aligned} \tag{2.50}$$

Non-oriented strings are defined by keeping only those states which are invariant under Ω. The resulting theories are insensitive to the orientation of the world-sheet. The vacuum is even under Ω, which is required from Ω to be unbroken by the interactions. For open strings the operation just eliminates the odd N states, and thus there is no massless state left: the gauge boson is projected out [see, however, in the section on charged strings that with these conditions some massless states may remain].

Occupation	Mass	State	Ω
$N=0$	$\alpha' M^2 = -1$	$\lvert k\rangle$	$+$
$N=1$	$\alpha' M^2 = 0$	$a_\mu \alpha_{-1}^\mu \lvert k\rangle$	$-$
$N=2$	$\alpha' M^2 = 1$	$b_{\mu\nu}\alpha_{-1}^\mu \alpha_{-1}^\nu \lvert k\rangle$	$+$
		$a_\mu \alpha_{-2}^\mu \lvert k\rangle$	$+$

(2.51)

For closed strings we obtain that only states which are left-right symmetric survive. At the massless level the antisymmetric tensor is projected out, whereas the graviton and dilaton are kept.

2.6 Charges

Open strings are allowed to carry charges at the end-points. These are known as Chan-Paton factors and give rise to non-abelian gauge groups of the type Sp(N) or O(N) in the unoriented case and U(N) in the oriented case. To see how this comes about, we will attach charges labeled by an index $i = 1, 2, \cdots, N$ at the two end-points of the open string. Then, the ground-state is labeled, apart from the momentum, by the end-point charges: $|k, i, j\rangle$, where i is on one end and j on the other. In the case of oriented strings, the massless states are $\alpha^{\mu}_{-1}|k, i, j\rangle$ and they give a collection of N^2 vectors. It can be shown that the gauge group is U(N) by studying the scattering amplitude of three vectors.

In the unoriented case, we will have to project by the transformation that interchanges the two string end-points Ω and also reverses the orientation of the string itself:

$$\Omega|k, i, j\rangle = \epsilon|k, j, i\rangle \,, \tag{2.52}$$

where $\epsilon^2 = 1$ since $\Omega^2 = 1$. Thus, from the N^2 massless vectors, only $N(N+1)/2$ survive when $\epsilon = 1$, forming the adjoint of Sp(N), while when $\epsilon = -1$, $N(N-1)/2$ survive, forming the adjoint of O(N).

3 Strings in perturbation theory

How do strings interact?
What is the result of those interactions for scattering of particles?
Which action effectively describes the particles of string theories?

3.1 Conformal symmetry

In the previous section, we have put rather little attention to conformal symmetry, though it is one of the major aspects of string theory. The Virasoro operators L_n and $\tilde{L}_n$ are in fact the generators of conformal symmetry on the 2-dimensional string worldsheet. Conformal symmetry is defined as the spacetime reparametrizations that leave angles invariant. E.g translations and rotations (the Poincaré group) are clearly part of it. But also dilations. In all dimensions $D > 2$ the full group of conformal transformations is $\mathrm{SO}(D, 2)$, of which $\mathrm{SO}(D-1, 1)$ are the Lorentz rotations, an extra $\mathrm{SO}(1, 1)$ is this dilation, and off the diagonal we find translations and some special conformal transformations. If we consider $D = 2$ then we see that this group splits in 2 parts as $\mathrm{SO}(2, 2) = \mathrm{SU}(1, 1) \times \mathrm{SU}(1, 1)$.

The 2 times 3 generators are L_{-1}, L_0, L_1 and $\tilde{L}_{-1}$, $\tilde{L}_0$, $\tilde{L}_1$. But in 2 dimensions the conformal group is infinite dimensional, and that is represented by the generators L_n and $\tilde{L}_n$. This can be understood as follows. The Polyakov action had GCT in 2 dimensions and dilatations, which act on the metric as

$$\delta h_{\alpha\beta} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + 2h_{\gamma(\alpha}\partial_{\beta)}\xi^\gamma + \Lambda_D \eta_{\alpha\beta} \,. \tag{3.1}$$

You can check that the conditions (2.12) determine Λ_D and allow $\partial_+\xi^- = \partial_-\xi^+ = 0$. I.e. arbitrary transformations of the form $\xi^+(\sigma^+)$ and $\xi^-(\sigma^-)$ remain possible and this contains the infinite number of parameters corresponding to L_n and $\tilde{L}_n$.

The condition on zero central charge in the Virasoro algebra is now a condition that this symmetry is non-anomalous.

Exercise 3.1: Expand $\xi^\pm(\sigma)$ in modes (e.g. for a closed string). Calculate the commutator of two conformal transformations, e.g. on the string coordinates with $\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$, and check that this gives rise to the classical Virasoro algebra.

3.2 Amplitudes from surfaces

From the perturbative point of view there is no essential difference between how you treat a field theory and a string theory. String theories just give rise to amplitudes that allow expansions where the coefficients A_g in (1.4) are given as sums of *surfaces* with g handles. The beauty of perturbative string theory, as contrasted with perturbative point-particle field theory, is that at fixed order g in the above expansion there is only one surface with g handles to consider, whereas the number of connected graphs grows as $g!$. So string theory simplifies the combinatorics, see figure 1 and figure 2.

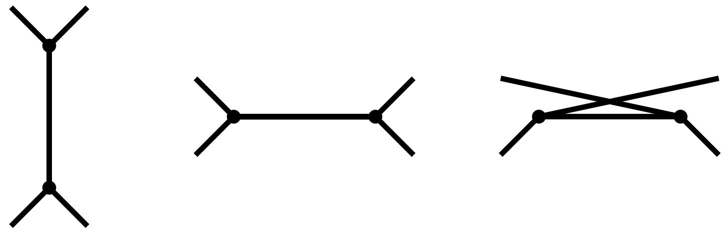

Figure 1: Feynman diagrams for a 4-point function for particles

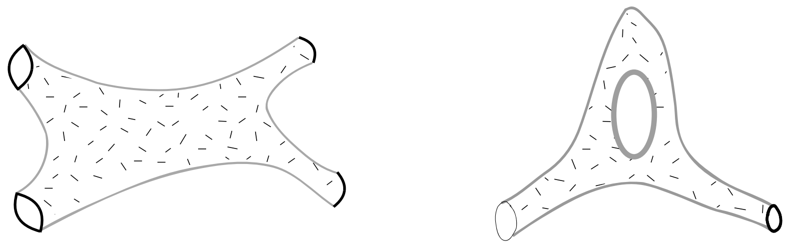

Figure 2: String interactions

We have to take into account, however, that a surface can be deformed. Topologically this does not change the surface, but the metric that describes this surface can change. Consider as simplest example a torus. Stretching the whole torus is a dilatation, which is part of the conformal symmetries already taken into account. However, we can still make one circle bigger, and keep the other invariant. It turns out that, apart from the conformal reparametrizations included in the infinitesimal generators L_n considered above, the inequivalent metrics are characterized by one complex parameter that we will call τ. Such a parameter is called a modulus.

In general the weight of a particular surface is computed as an integral over the moduli space of inequivalent conformal metrics $\mathcal{M}_g$ on the surface

$$A_g = \int_{\mathcal{M}_g} w_g \,. \tag{3.2}$$

To make this more concrete, let us return to the path integral (1.3). First, we make a technical step. We perform a complex rotation to go to Euclidean space, rather than Minkowski. This is a rotation on some of the fields in complex space (leading to $h_{00} \to -h_{00}$). The result is that we may use on the 2-dimensional space an Euclidean metric, which makes all surfaces much better behaved.

We consider now scattering of M string states. We still have to define how we relate different scattering processes. In field theory, we had a coupling constant g, and any vertex contained such a factor. Here, we will associate an a priori undetermined factor κ to any split of the world surface. E.g. the scattering of 3 string states without a 'loop' will be proportional to κ. Furthermore any extra loop involves a splitting and joining of the world surface, which brings in a factor κ^2. Therefore we will associate a factor

$$\kappa^{M-2+2g} = \kappa^{M-\chi}\,, \qquad \chi \equiv 2 - 2g\,, \tag{3.3}$$

to any worldsheet, where g is the genus of the surface (neglecting external states), and χ is the Euler number. E.g. for a 2-sphere, the genus is 0 and the Euler number is 2, while for a torus, the genus is 1, and the Euler number is 0.

Therefore, the path integral can be written as

$$\begin{aligned} \langle \mathcal{A}(1,\ldots,M)\rangle &= \frac{1}{Z}\sum_{g=0}^{\infty}\kappa^{M-\chi}N_g\int \mathcal{D}X^\mu \mathcal{D}h_{\alpha\beta}\,\mathcal{A}(1,\ldots,M)\,\mathrm{e}^{-S}\,, \\ S &= \frac{1}{4\pi\alpha'}\int_{\Sigma(g)} \mathrm{d}^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\,, \end{aligned} \tag{3.4}$$

where Z is the normalizing factor (the same integral without the $\mathcal{A}$ factor).

The factor N_g is a normalization factor that should take into account that we integrate over gauge-equivalent metrics. We can formally write $N_g^{-1} = V_{GCT\times dil}$, the field volume for the transformations in the gauge group. This volume is just a number that can be absorbed in a normalization constant as long as the invariances are still present in the quantum theory. This is the anomaly-free condition that restricts us to $D = 26$.

$\Sigma(g)$ is the world surface of genus g, and $h_{\alpha\beta}$ is the Euclidean metric, for which we can locally impose the gauge

$$h_{\alpha\beta} = \delta_{\alpha\beta}\,. \tag{3.5}$$

To do so in a correct way (taking into account the measure of the path integral), we should use ghosts and antighosts as mentioned before. But we are not going in these technical aspects here.

Though we can locally choose (3.5), this gauge cannot be imposed globally. what is more important is that this does not take into account all the reparametrization equivalences of the metric, but only those that are continuously connected to the identity. This is the issue of moduli mentioned above. All that can be achieved is to map $h_{\alpha\beta}$ to a metric of constant curvature,

$$h_{\alpha\beta} \stackrel{!}{=} \hat{h}_{\alpha\beta}[\vec{\tau}]\,. \tag{3.6}$$

$\Sigma(g)$ in general possesses a continuous family of such metrics, parametrized by moduli $\vec{\tau} = (\tau_1,\ldots)$. The space of constant curvature metrics on a two-dimensional closed compact surface is isomorphic to the space of complex structures. By reparametrizations and Weyl transformations we cannot change the complex structure of the metric but we can map it to the unique representative (3.6) of the complex structure class which has constant curvature. Then the path integral over all metrics reduces to a finite-dimensional integral over the space $\mathcal{M}_g$ of complex structures. The dimension of this space is known from a Riemann-Roch theorem. For $g = 0$ (the sphere) the complex structure is unique, and every metric can be mapped to the standard round metric on the sphere. For $g \geq 1$ there is a non-trivial moduli space,

$$\begin{aligned} \dim_{\mathbf{C}} \mathcal{M}_g &= 1\,, \qquad \text{for } g = 1\,, \\ \dim_{\mathbf{C}} \mathcal{M}_g &= 3g - 3\,, \quad \text{for } g > 1\,. \end{aligned} \tag{3.7}$$

The one (complex) variable for the torus is the one that characterizes the length of the two commuting circles. We will make this explicit below.

After carrying out the integration over the metric, amplitudes take the form

$$\langle \mathcal{A}(1,\ldots,M)\rangle = \frac{1}{Z}\sum_{g=0}^{\infty} \kappa^{M-\chi} N_g' \int_{\mathcal{M}_g} \mathrm{d}\mu(\vec{\tau}) \int \mathcal{D}X^\mu \mathrm{e}^{-S[X,\hat{h}]} J(\hat{h}) \mathcal{A}(1,\ldots,M)\,. \tag{3.8}$$

N_g' are normalization factors needed to deal with the X^μ-integration and $J(\hat{h})$ is the Faddeev-Popov determinant, which one can rewrite as a functional integral over Faddeev-Popov ghost fields. As indicated the X^μ-integral depends on the moduli through the world-sheet metric $\hat{h}_{\alpha\beta} = \hat{h}_{\alpha\beta}(\vec{\tau})$. The measure $\mathrm{d}\mu(\vec{\tau})$ for the moduli is the natural measure on the space of complex structures, the so-called Weil-Petersson measure. It is not a trivial issue to determine the metric to be used. The measure should satisfy certain symmetry properties. This will become clear with the example of the torus.

3.3 Example of the torus

As a simple and useful illustration, consider the one-loop vacuum to vacuum string amplitude, see figure 3.

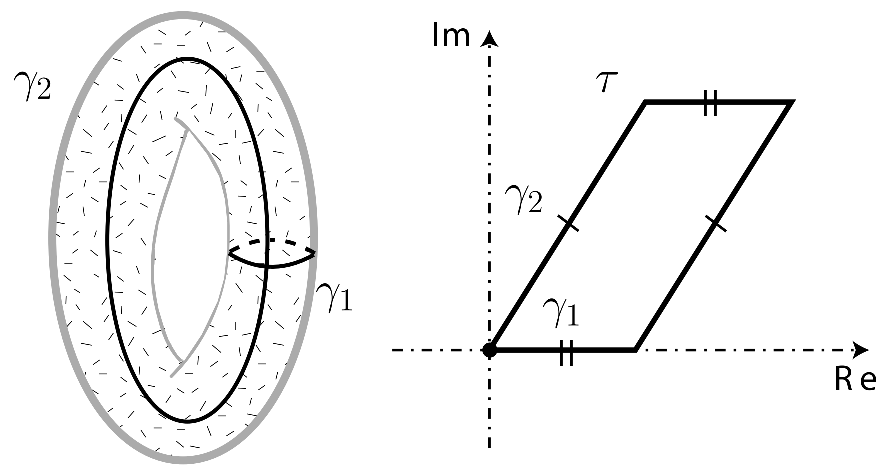

Figure 3: One loop vacuum to vacuum amplitude and the torus lattice.

This has the physical interpretation of calculating the vacuum energy. There is no external string (so we are in fact calculating Z) and the worldsheet is topologically a torus.

By using conformal symmetry we can pick a constant metric so that the volume is normalized to 1. Pick coordinates $\sigma^1, \sigma^2 \in [0,1]$. Then the volume is 1 if the determinant of the metric is 1. We can parametrize the metric, which is also a symmetric and positive-definite matrix, by a single complex number $\tau = \tau_1 + \mathrm{i}\tau_2$, with positive imaginary part $\tau_2 \geq 0$ as follows:

$$h_{\alpha\beta} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} . \tag{3.9}$$

The line element is

$$\mathrm{d}s^2 = h_{\alpha\beta}\mathrm{d}\sigma^\alpha \mathrm{d}\sigma^\beta = \frac{1}{\tau_2}\left|\mathrm{d}\sigma^1 + \tau \mathrm{d}\sigma^2\right|^2 = \frac{\mathrm{d}w\mathrm{d}\bar{w}}{\tau_2} , \tag{3.10}$$

where

$$w = \sigma^1 + \tau\sigma^2 , \qquad \bar{w} = \sigma^1 + \bar{\tau}\sigma^2 \tag{3.11}$$

are the complex coordinates of the torus. This is the reason why the parameter τ is known as the complex structure (or modulus) of the torus. It cannot be changed by infinitesimal diffeomorphisms or Weyl rescalings and is thus the complex 'Teichmüller' parameter of the torus. The periodicity properties of σ^1, σ^2 translate to

$$w \to w + 1 , \qquad w \to w + \tau . \tag{3.12}$$

The torus can be thought of as the points of the complex plane w identified under two translation vectors corresponding to the complex numbers 1 and τ. The integration over $\mathcal{M}_g$ now reduces to an integration over the moduli parameter τ, and the factor in (3.10) implies that we will integrate with measure $\int \frac{\mathrm{d}^2\tau}{\tau_2}$. However, there are still some τ that are equivalent, see below. Non-degeneracy of the torus requires $\mathrm{Im}\,\tau \neq 0$, and by choices of basis of lattice vector we can require τ to live on the upper complex half-plane. Let us look at this from the worldsheet viewpoint.

There is an interesting construction of the torus starting from the cylinder. Consider a cylinder of length $2\pi\tau_2$ and circumference 1. Take one end, rotate it by an angle $2\pi\tau_1$ and glue it to the other end. This produces a torus with modulus $\tau = \tau_1 + \mathrm{i}\tau_2$. This construction gives a very useful relation between the path integral of a CFT on the torus and a trace over the Hilbert space. Choose the imaginary axis as worldsheet 'time' and real axis as the spatial extent of the string. First, the propagation along the cylinder is governed by the hamiltonian (generator of time translations) $H = 2(L_0 - 1 + \tilde{L}_0 - 1)$. In the classical theory, we found $2(L_0 + \tilde{L}_0)$, see (2.23). But then L_0 was not yet well defined due to normal ordering. As the vacuum, which is time invariant, satisfies $(L_0 - 1)|\mathrm{vac}\rangle = 0$, we see that we have to use $L_0 - 1$. The rotation around the cylinder is implemented by the 'momentum' operator $P = 2L_0 - 2\tilde{L}_0$, see (2.23). The normalization of the string was taken such that σ evolves with π over the length of the string (which is the circumference of the cilinder normalized to 1 above). Therefore the rotation with angle $2\pi\tau_1$ mentioned above, gives a translation in σ over $\pi\tau_1$.

As there is no end to the string in this one-loop amplitude, the path integral sums over all states in the Hilbert space — it is a trace. In fact it is the partition function

$$\begin{aligned} Z &= \int \frac{\mathrm{d}^2\tau}{\tau_2}\,\mathrm{Tr}\left[\exp\left(-\pi\tau_2 H + \mathrm{i}\pi\tau_1 P\right)\right] \\ &= \int \frac{\mathrm{d}^2\tau}{\tau_2}\,\mathrm{Tr}\left[\exp\left(-2\pi\tau_2(L_0+\tilde{L}_0-2) + 2\mathrm{i}\pi\tau_1(L_0-\tilde{L}_0)\right)\right] \\ &= \int \frac{\mathrm{d}^2\tau}{\tau_2}\,\mathrm{Tr}\left[q^{L_0-1}\bar{q}^{\tilde{L}_0-1}\right] \qquad \text{with } q = \mathrm{e}^{2\pi\mathrm{i}\tau} \\ &= \int \frac{\mathrm{d}^2\tau}{\tau_2}(q\bar{q})^{-1}\left|\prod_{n=1}^{\infty}(1-q^n)^{-24}\right|^2 V_{26}\int\frac{\mathrm{d}^{26}k}{(2\pi)^{26}}\,\mathrm{e}^{-\pi\alpha'\tau_2 k^2} \\ &= V_{26}\alpha'^{-13}\int\frac{\mathrm{d}^2\tau}{\tau_2^2}\left(\sqrt{\tau_2}\eta\bar{\eta}\right)^{-24} \qquad \text{with } \eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \end{aligned} \tag{3.13}$$

When performing the trace, we recognize the sum (2.45) for the parts N and $\tilde{N}$ of L_0 and $\tilde{L}_0$. Remember that states in the loop are not 'on-shell', so $L_0 \neq 1$. We then still have to consider the trace over momenta, which, with some continuum normalization of the momenta integral over the momenta becomes $V_D\int\frac{\mathrm{d}k^D}{(2\pi)^D}$, where V_D is the volume of the D-dimensional space. Finally we use $\int\frac{\mathrm{d}k^D}{(2\pi)^D}\mathrm{e}^{-\pi a k^2} = a^{-D/2}$. You have probably noticed that this trace, weighted by the exponential of the hamiltonian and other conserved quantities is a partition function in the terminology in statistical mechanics.

To complete the calculation of the amplitude, one also needs to integrate over the moduli parameter τ, which parameterizes the length and twist of the torus as discussed earlier. Observe that the integration over $\mathrm{Re}\,\tau$ enforces the level matching condition, as those terms with unequal exponents for q and $\bar{q}$ will vanish.

3.4 Modular transformations

The function $\eta(q)$ is called the Dedekind's function, which satisfies

$$\eta(-\tau^{-1}) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau)\,. \tag{3.14}$$

As $\tau_2 \to \tau_2/|\tau|^2$ under $\tau \to -1/\tau$, the quantity in brackets in the final expression is invariant under this transformation. Also the integration including the τ_2^{-2} is invariant. Further, it is obvious that the result is invariant under $\tau \to \tau+1$.

These two invariance transformations are essential. In fact, it is easy to see that two tori with parameter τ related by such transformations are in fact the same tori in different descriptions (draw the lattices). The two transformations

$$\tau \to \tau+1\,, \qquad \tau \to -\frac{1}{\tau}\,, \tag{3.15}$$

generate a full group

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}\,, \qquad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{S}\,\ell(2,\mathbb{Z})\,. \tag{3.16}$$

Exercise 3.2: Show that this invariance is implied by the two generic transformations.

As the tori are the same, the invariance under these transformations is a consistency condition, known as '*modular invariance*'. This condition becomes non-trivial when considering more general background geometries or string theories with fermions.

The moduli space is obtained by restricting τ to a fundamental domain $\mathcal{F}$ of the action of $\mathrm{S}\,\ell(2,\mathbb{Z})$, i.e. a domain such that all other values of τ are related by these transformations. A convenient choice is

$$\mathcal{F} = \left\{ \mathrm{Im}\,\tau > 0,\ -\tfrac{1}{2} \leq \mathrm{Re}\,\tau < \tfrac{1}{2},\ |\tau| \geq 1 \right\}. \tag{3.17}$$

Exercise 3.3: Show that all other regions can be mapped to this one.

Modular invariance has deep consequences for the short distance behaviour of string theory. In fact, modular invariance is what makes closed string theories UV finite. To illustrate how this works, let us represent the one-loop amplitude in closed string theory as (this could be with sources, but is similar to what we saw for Z)

$$A^{\text{string}}_{1-\text{loop}} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^2} F(\tau)\,. \tag{3.18}$$

An analogous expression for one loop amplitudes in quantum field theory is given by the proper time parametrization of Schwinger:

$$A^{\text{QFT}}_{1-\text{loop}} = \int_{\varepsilon}^{\infty} \frac{\mathrm{d}t}{t} f(t)\,, \tag{3.19}$$

where t is the proper time and ε is an UV cutoff. In this formulation UV divergences occur at short times $t \to 0$. In string theory $\mathrm{Im}\,\tau$ plays the role of proper time, and potential UV divergences occur for $\mathrm{Im}\,\tau \to 0$. However, by restricting to the fundamental domain we have cut out the whole dangerous region of small times and high momenta. This confirms the intuitive idea that strings should have a particularly soft UV behaviour, because the theory has a minimal length scale, which works like a physical UV cutoff.

3.5 Vertex operators

In the example above, there were no external states. External states are included by taking specific operators as $\mathcal{A}$ in (3.4). In general these operators are of the form

$$V(\Phi) = \int_{\Sigma_g} \mathrm{d}^2\sigma \sqrt{h} V_\Phi(\sigma)\,, \tag{3.20}$$

obtained by integrating the vertex operators $V(\sigma)$ over the world sheet. These vertex operators should still respect the conformal transformations $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$ of the worldsheet. In conformal field theory fields that transform covariantly under conformal transformations are called primary conformal fields. A primary conformal field of weights $(h, \overline{h})$ is an object that transforms like a contravariant tensor field of rank $(h, \overline{h})$:

$$\tilde{V}(\tilde{\sigma}^+, \tilde{\sigma}^-) = \left(\frac{\mathrm{d}\sigma^+}{\mathrm{d}\tilde{\sigma}^+}\right)^h \left(\frac{\mathrm{d}\sigma^-}{\mathrm{d}\tilde{\sigma}^-}\right)^{\overline{h}} V(\sigma^+, \sigma^-)\,. \tag{3.21}$$

It is clear that in order to give an invariant integrating over $\int \mathrm{d}\sigma^+ \mathrm{d}\sigma^-$ that we need $h = \overline{h} = 1$. This property is equivalent to imposing the Virasoro constraints (2.35) on physical states. Incoming states are constructed from vertex operators by applying them to a ground state $|0\rangle$ and taking the limit to the infinite past $\sigma^0 \to -\infty$.

$$|\Phi\rangle = \lim_{\sigma^0 \to -\infty} V_\Phi(\sigma)|0\rangle\,, \tag{3.22}$$

where $|0\rangle$ is the $|k = 0\rangle$ (unphysical) zero-momentum state with occupation numbers $N = 0 = \tilde{N}$.

Thus there is such a vertex operator for any physical state. E.g. the tachyon corresponds to

$$V_{\text{tachyon}}(\sigma) =: \mathrm{e}^{\mathrm{i}k_\mu X^\mu} : (\sigma)\,, \tag{3.23}$$

where $: \cdots :$ indicates normal ordering (α_n for $n > 0$ to the right).

Exercise 3.4: (Advanced). Consider how these statements can be expressed in Operator product expansions. You can find the necessary ingredients in sections 2.1-4 of the book of Polchinski.

The corresponding state $V_{\text{tachyon}}(\sigma)|0\rangle$ in the limit $\sigma^0 \to -\infty$ describes an incoming tachyon.

Exercise 3.5: Check that

$$\lim_{\sigma^0 \to -\infty} : \mathrm{e}^{\mathrm{i}k_\mu X^\mu} : (\sigma)|0\rangle = \mathrm{e}^{\mathrm{i}k_\mu x^\mu}|0\rangle = |k\rangle\,. \tag{3.24}$$

Without going in details (those knowing conformal field theory techniques may check this), we find that the conformal weights $(h, \overline{h})$ of some primary operators are

$$\partial_+ X^\mu \;\; (1, 0)\,, \qquad \partial_- X^\mu \;\; (0, 1)\,, \qquad : \mathrm{e}^{\mathrm{i}k_\mu X^\mu} : \;\; (\tfrac{1}{4}\alpha' k^2, \tfrac{1}{4}\alpha' k^2)\,. \tag{3.25}$$

This shows thus that the tachyon indeed has $\alpha' M^2 = -4$ as we found before.

The massless states are produced by the vertex operators

$$V_0(\zeta) = \zeta_{\mu\nu} : \partial_+ X^\mu \partial_- X^\nu \mathrm{e}^{\mathrm{i} k_\mu X^\mu} : , \tag{3.26}$$

which has weights $(1,1)$ if

$$k^2 = 0 , \qquad k^\mu \zeta_{\mu\nu} = k^\nu \zeta_{\mu\nu} = 0 , \tag{3.27}$$

in agreement with the conditions that we found before for the states (2.47).

3.6 Effective actions

One can then calculate the scattering of states. We do not explain the calculations here, but will comment on a very important result: the scattering of 2 states with vertex operators (3.26) for symmetric $\zeta_{\mu\nu}$ to two outgoing states of the same kind. Assigning momenta $k^{(i)}$ for $i = 1, \ldots, 4$ to these 4 states, and defining the Mandelstam variables

$$s = -(k^{(1)} + k^{(2)})^2 , \qquad t = -(k^{(2)} + k^{(3)})^2 , \qquad u = -(k^{(1)} + k^{(3)})^2 , \tag{3.28}$$

one finds

$$A_4^{\text{String}} = \frac{\Gamma(1 - \frac{\alpha'}{4} s)\Gamma(1 - \frac{\alpha'}{4} t)\Gamma(1 - \frac{\alpha'}{4} u)}{\Gamma(1 + \frac{\alpha'}{4} s)\Gamma(1 + \frac{\alpha'}{4} t)\Gamma(1 + \frac{\alpha'}{4} u)} A_4^{\text{FTh}} , \tag{3.29}$$

where A_4^{FTh} is a result from field theory. It is the amplitude for scattering of 2 gravitons using the Einstein action (1.8)

$$S_{\text{GR}} = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{g} R(g) , \tag{3.30}$$

for $D = 26$ and using the expansion

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa \tilde{\zeta}_{\mu\nu}(x) , \tag{3.31}$$

where $\zeta_{\mu\nu}(k)$ and $\tilde{\zeta}_{\mu\nu}(x)$ are each others Fourier transforms. We thus obtain that the lowest order in α' of this string amplitude agrees with general relativity. In general, one can find effective field theories that give the same scattering amplitudes as the ones obtained in field theory order by order in α'. We can continue matching the string amplitude using

$$S_{\text{eff}} = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{g} \left(R(g) + \alpha' c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \cdots + \mathcal{O}((\alpha')^2) \right) , \tag{3.32}$$

where c_1 is a numerical constant. The α'-expansion of the effective action is an expansion in derivatives. It is valid at low energies, i.e., at energies lower than the scale set by α', where corrections due to massive string scales are small.

Obviously, it is very cumbersome to construct the effective action by matching field theory amplitudes with string amplitudes. In practice one uses symmetries to constrain the form of the effective action. This is particularly efficient for supersymmetric actions, which only depend on a few independent parameters or functions, which can be extracted from a small number of string amplitudes. A different technique, which often is even more efficient, is to study strings in curved backgrounds, and, more generally, in background fields.

3.7 Strings in curved backgrounds

So far we only discussed strings in flat backgrounds. Let us now consider the case of a curved background with Riemannian metric $G_{\mu\nu}(X)$. Then the Polyakov action takes the form of a non-linear sigma-model

$$S_{\mathrm{P}}(G) = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)\,. \tag{3.33}$$

As emphasized above, the local Weyl invariance

$$h_{\alpha\beta} \to \mathrm{e}^{\lambda(\sigma)} h_{\alpha\beta}\,, \qquad \text{or infinitesimal} \qquad \delta h_{\alpha\beta} = \lambda h_{\alpha\beta}\,, \tag{3.34}$$

is crucial for the consistency of string theory, since the construction of states, vertex operators and amplitudes is based on having a conformal field theory on the world-sheet. It is expressed as the vanishing of the trace of the energy-momentum tensor. The latter is defined as

$$T^{\alpha\beta} = \frac{4\pi}{\sqrt{h}}\frac{\delta S}{\delta h_{\alpha\beta}}\,. \tag{3.35}$$

Thus the Weyl invariance implies $T^{\alpha\beta}h_{\alpha\beta} = 0$, which in our coordinates is $T^{+-} = 0$. If the spacetime metric is curved, then the Weyl invariance of the classical action (3.33) is still manifest. But at the quantum level it becomes non-trivial and imposes restrictions on $G_{\mu\nu}(X)$. The requirement is now that $\langle T^{\alpha\beta}h_{\alpha\beta}\rangle = 0$. Calculations (which we will not discuss here) reveal that

$$\langle T^{\alpha\beta}h_{\alpha\beta}\rangle = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu\,, \tag{3.36}$$

with

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + \tfrac{1}{2}\alpha'^2 R_{\mu\rho\sigma\tau}R_\nu{}^{\rho\sigma\tau} + \mathcal{O}(\alpha'^3)\,. \tag{3.37}$$

Thus, we find again that this is zero if the Einstein field equations in first order of α' are used, or more general the field equations from (3.32). This gives another argument that in this order of α' the action (3.32) is the effective particle description for these string modes.

We can relate the two arguments in the following way. If we expand again

$$G_{\mu\nu} = \eta_{\mu\nu} + \kappa\tilde{\zeta}_{\mu\nu}\,, \tag{3.38}$$

then the action (3.33) expands as

$$S_P[G_{\mu\nu}] = S_P[\eta_{\mu\nu}] + \kappa V[\tilde{\zeta}_{\mu\nu}]\,, \tag{3.39}$$

where

$$V[\tilde{\zeta}_{\mu\nu}] = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{h}h^{\alpha\beta}\tilde{\zeta}_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu\,. \tag{3.40}$$

Taking the Fourier transform of $\psi_{\mu\nu}(X)$ we obtain

$$V[\tilde{\zeta}_{\mu\nu}] = \frac{1}{4\pi\alpha'}\int \mathrm{d}^D k\int \mathrm{d}^2\sigma\sqrt{h}V(k,\zeta(k))\,, \tag{3.41}$$

where

$$V(k,\zeta(k)) = \zeta_{\mu\nu}(k)\partial_\alpha X^\mu \partial^\alpha X^\nu \mathrm{e}^{\mathrm{i}k_\rho X^\rho} \tag{3.42}$$

is the graviton vertex operator as in (3.26) (with symmetric and traceless $\zeta_{\mu\nu}$). Thus in the path integral, the expansion in κ is the same as inserting vertex operators creating gravitons.

Then it is straightforward to include also the other massless modes: the antisymmetric tensor and the dilaton. Indeed, we can generalize the previous approach and start with

$$S_\sigma = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left\{ \left[\sqrt{h}h^{\alpha\beta}G_{\mu\nu}(X) + \mathrm{i}\epsilon^{\alpha\beta}B_{\mu\nu}(X)\right]\partial_\alpha X^\mu \partial_\beta X^\nu + \alpha'\Phi\sqrt{h}R^{(2)}(h)\right\}\,, \tag{3.43}$$

where $B_{\mu\nu}$ is the background antisymmetric tensor field and Φ is the background value of the dilaton. The last term is for later convenience (normalization of Φ) written with an extra factor α'. We will come back to the meaning of this term. The next step is to do a full analysis of this new action and ensure that in the quantum theory, one has Weyl invariance, which amounts to the tracelessness of the two dimensional stress tensor. The calculations give now

$$\langle T^{\alpha\beta}h_{\alpha\beta}\rangle = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{\mathrm{i}}{2\alpha'}\beta^B_{\mu\nu}h^{-1/2}\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - \frac{1}{2}\beta^\Phi(X)R^{(2)(h)}\,. \tag{3.44}$$

with

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\partial_\nu\Phi - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}\right) + \mathcal{O}(\alpha'^2),\\
\beta^B_{\mu\nu} &= \alpha'\left(-\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + \nabla^\rho\Phi H_{\rho\mu\nu}\right) + \mathcal{O}(\alpha'^2),\\
\beta^\Phi &= \alpha'\left(\frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + \mathcal{O}(\alpha'^2)\,,
\end{aligned} \tag{3.45}$$

with $H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. For Weyl invariance, we ask that each of these beta functions for the sigma model couplings actually vanish. We remark at the level α'^0 the requirement $D=26$. The remarkable thing is that these resemble *spacetime field equations for the background fields.* In fact, the field equations can be derived from the following spacetime action:

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^D X\sqrt{G}\mathrm{e}^{-2\Phi}\left[R(G) + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \mathcal{O}(\alpha')\right]\,. \tag{3.46}$$

3.8 Coupling constants and frames

We first comment on the last term of (3.43). If Φ is a constant, then the integrand of that term is a total derivative, and depends on the global properties of the string worldsheet. The value is related to the Euler number that we mentioned in (3.3). In fact, we have

$$\chi = \frac{1}{4\pi}\int_{\Sigma(g)} \mathrm{d}^2\sigma\sqrt{h}R^{(2)}(h) = 2-2g\,. \tag{3.47}$$

Hence, if we shift Φ by a constant, the action changes proportional to the Euler number, and this gives a factor in the path integral:

$$\begin{array}{rcl}\Phi \rightarrow \Phi + a & \Rightarrow & S \rightarrow S + a\,\chi \\ & & \kappa^{-\chi}\mathrm{e}^{-S} \rightarrow (\kappa\mathrm{e}^{-a})^{-\chi}\mathrm{e}^{-S}\,.\end{array} \tag{3.48}$$

Hence we see that this is absorbed in a redefinition of the coupling constant κ. Hence, κ is not an independent parameter! It can be chosen at random by changing the value of the dilaton.

In (3.46) we wrote an overall coupling constant κ_0 rather than κ because the Einstein term is not yet canonically normalized. If you did the last part of exercise 1.4, you know how to solve this. When we define

$$g_{\mu\nu} \equiv G_{\mu\nu}\exp\left[-\frac{4}{D-2}(\Phi-\Phi_0)\right]\,, \tag{3.49}$$

then

$$\int \mathrm{d}^D x\,\sqrt{G}\mathrm{e}^{-2(\Phi-\Phi_0)}R(G) = \int \mathrm{d}^D x\,\sqrt{g}\left[R(g) - 4\frac{D-1}{D-2}\partial_\mu\Phi\partial^\mu\Phi\right]. \tag{3.50}$$

Therefore the action ‘in string frame’ (3.46) can be rewritten ‘in Einstein frame’ as

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D X\sqrt{g}\left[R(g) - \frac{4}{D-2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}\mathrm{e}^{-\frac{8}{D-2}(\Phi-\Phi_0)}H_{\mu\nu\rho}H^{\mu\nu\rho} + \mathcal{O}(\alpha')\right], \tag{3.51}$$

when we define

$$\kappa = \kappa_0\mathrm{e}^{\Phi_0}\,. \tag{3.52}$$

We can identify the ‘string coupling constant’ as

$$g_S = \mathrm{e}^{\Phi_0}\,, \qquad \kappa = \kappa_0 g_S\,. \tag{3.53}$$

So far, there is no potential for the dilaton, and its vacuum expectation value is thus a free parameter labeling different ground states. The only scale that is in the game is α' and that should provide the gravitational coupling constant. Hence this gives that α' is of the order of the Planck mass squared, and massive modes of string theory are really very high mass. Therefore, for many purposes they can be neglected.

For practical applications, both the string frame effective action and the Einstein frame effective action (and their higher-loop generalizations) are needed. The string frame action is adapted to string perturbation theory and has a universal dependence on the dilaton and on the string coupling. As the string coupling constant appears as an overall factor in the action, it plays the role like $\hbar$ in (1.2) and counts the loops:

$$S^{\mathrm{StrFr}}_{g-\mathrm{loop}} \sim g_S^{-2+2g}\,. \tag{3.54}$$

In the Einstein frame, gravity and the scalar are decoupled in the kinetic energy. Therefore in this frame, the metric describes the physical gravitational field. Hence, this action is needed when analyzing gravitational physics, in particular for solutions of the effective action that describe black holes and other spacetime geometries.

3.9 Open strings

Finally let us comment on *open strings.* As you may imagine intuitively, open strings may have interactions producing closed strings. Therefore consistency of open string theories at the quantum level requires the inclusion of closed strings. This means in particular that *every consistent quantum string theory has to include gravity.* The relation between open and closed strings becomes obvious when one realizes that the annulus is topologically equivalent to the cylinder. While the annulus is the open string one loop diagram, the cylinder is the closed string propagator.

We may also write the effective action which summarizes the leading order (in α') open string physics at tree level:

$$S_{\text{eff,open}} = -\frac{\mathrm{C}}{4}\int \mathrm{d}^D X\, \mathrm{e}^{-\Phi}\, \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \mathcal{O}(\alpha')\,, \tag{3.55}$$

with C a dimensionful constant which we will fix later. It is of course of the form of the Yang–Mills action, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field A_μ is coupled in sigma-model fashion to the boundary of the world sheet by the boundary action:

$$\int_{\partial\mathcal{M}} \mathrm{d}\tau\, A_\mu \partial_t X^\mu\,. \tag{3.56}$$

It turns out that the coupling constant for open string amplitudes is the square root of the one for closed strings, as we see also in the overall factor $\mathrm{e}^{-\Phi}$ in (3.55).

For closed strings the UV finiteness was guaranteed by modular invariance. For theories with open and closed strings there is another crucial property, which is tadpole cancellation. It turns out that this can only be realized for non-oriented open strings when the gauge group is $\mathrm{SO}(2^{D/2}) = \mathrm{SO}(8192)$. However, it is time to go to something more realistic.

4 Superstring theories

Why do we need supersymmetry?
How many dimensions can there be in a consistent field theory?
How many string theories are there?
What is the content of the main classical string theories?

Nature has bosons and fermions. We should take this into account, and will find the bonus that we can get rid of the tachyon.

There are 2 main approaches: Ramond-Neveu-Schwarz (RNS) and Green-Schwarz (GS). The latter is technically more involved, and we will not discuss it here, though it is a good approach to have covariant string actions and to make the bridge to branes.

4.1 Action

In the RNS formalism, we first introduce new fields ψ_A^μ, which are spinors on the worldsheet, for which A is the index running over 2 values, while they are vectors in spacetime, represented by the index μ that runs as before over $\mu = 0, 1 \ldots, D-1$.

In a first step, after introducing these fermions (defining an action, solutions with boundary conditions, ...) we still have tachyons. However, then it will turn out that one can define a projection, GSO (Gliozzi, Scherk, Olive) such that the tachyon is projected out. It turns out that this projection is necessary at the quantum level (for modular invariance). It turns out that there are different possibilities leading to different superstring theories.

Supersymmetry can be seen as a fermionic version of general coordinate transformations. The analogue of the conformal group is a 'superconformal group'. We saw that in the bosonic case, the conformal group has an infinite number of generators, represented by the L_n. The fermionic partners are also an infinite number of generators.

In the bosonic case we introduced first a metric on the worldsheet $h_{\alpha\beta}$ and then fixed it to $\eta_{\alpha\beta}$, but were left with constraints that were $++$ and $--$ components of the energy momentum tensors. Here we should in principle first also introduce a partner for the metric, a 'gravitino', but in the same way as the graviton it has no physical degrees of freedom. It can be fixed, and we are left with extra constraints.

First the technical tools: the Clifford algebra. The place of gamma matrices is taken here by the two matrices

$$\rho_0 = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_* = \rho_0\rho_1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.1)$$

The former two satisfy $\rho_\alpha\rho_\beta + \rho_\beta\rho_\alpha = 2\eta_{\alpha\beta}$. The latter is similar to the γ_5 matrix in 4 dimensions. In 2 dimensions, contrary to 4 dimensions, it is possible to choose chiral spinors that are real. The projection operators are canonically $(\mathbb{1} \pm \rho_*)/2$, but with the chosen basis this is just selecting the upper or the lower component of the spinor.

The action after eliminating the metric and the gravitino is

$$S_{\text{RNS}} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + \overline{\psi}^\mu \partial\!\!\!/ \psi_\mu\right). \tag{4.2}$$

In 2 dimensions, $\overline{\psi} = \psi^T \rho_0$. The action is real if we define complex conjugation on spinors without interchange of their position. The action is supersymmetric for transformations with (constant) parameter ϵ:

$$\delta X^\mu = \bar\epsilon \psi^\mu\,, \qquad \delta\psi_\mu = \partial\!\!\!/ X^\mu \epsilon \;\rightarrow\; \delta\overline{\psi}^\mu = -\bar\epsilon \partial\!\!\!/ X^\mu\,. \tag{4.3}$$

The fermionic part of the action can be written simply in terms of the upper and lower components of the spinor, which we denote respectively as ψ_-^μ and $\psi_+\mu$ as

$$\overline{\psi}^\mu \partial\!\!\!/ \psi_\mu = \psi_-^\mu(\partial_0 + \partial_1)\psi_{-\mu} + \psi_+^\mu(\partial_0 - \partial_1)\psi_{+\mu}\,. \tag{4.4}$$

The Euler-Lagrange equations lead then to the splitting of the dependence of the spinors on the light-cone coordinates:

$$\psi_+^\mu = \psi_+^\mu(\sigma^+)\,, \qquad \psi_-^\mu = \psi_-^\mu(\sigma^-)\,. \tag{4.5}$$

4.2 R and NS boundary conditions

But to arrive at this result we have to be able to remove the boundary terms when integrating over σ. This is thus related to the ∂_1 terms in (4.4), and the condition is that the variations satisfy

$$\left(\psi_-^\mu \delta\psi_{-\mu} - \psi_+^\mu \delta\psi_{+\mu}\right)\Big|_{\sigma=0}^{\sigma=\pi} = 0\,. \tag{4.6}$$

There are again two types of solutions: those were the values of ψ at $\sigma = 0$ and $\sigma = \pi$ are related are closed string modes, while if the conditions are separately satisfied, then we have open string modes. Let us start with the latter. To satisfy (4.6) at $\sigma = 0$, we can have $\psi_+(0) = \pm\psi_-(0)$. However, so far, ψ_+ and ψ_- are independent variables and we can still define $\psi'_- = -\psi_-$ without any consequence. So we fix this freedom by choosing $\psi_+(0) = \psi_-(0)$. Then they are fixed and we have two different choices at $\sigma = \pi$. These two choices go under the name Ramond (R) and Neveu-Schwarz (NS) boundary conditions. Thus[2]

$$\psi_+(0) = \psi_-(0)\,, \qquad \text{and} \qquad \psi_+(\pi) = \left\{\begin{array}{ll} \psi_-(\pi) & R \\ -\psi_-(\pi) & NS \end{array}\right. \tag{4.7}$$

For closed string modes, the ψ_+ and ψ_- are each coupled at $\sigma = \pi$ to their value at $\sigma = 0$. There are 2 choices for each and these are denoted again by R or NS:

$$\psi_+^\mu(\pi) = \left\{\begin{array}{ll} \psi_+(0) & R \\ -\psi_+(0) & NS \end{array}\right. \qquad \text{and} \qquad \psi_-^\mu(\pi) = \left\{\begin{array}{ll} \psi_-(0) & R \\ -\psi_-(0) & NS \end{array}\right. \tag{4.8}$$

[2]We indicate only the dependence on σ^1. These equations should hold at any time σ^0.

This gives thus R-R, NS-R, R-NS or NS-NS modes. To find these modes, we have to define an expansion as we did for the bosonic case. We define

$$\begin{array}{llll}
\text{open}: & \psi^\mu_\pm = \dfrac{1}{\sqrt{2}}\sum_r \psi^\mu_r \mathrm{e}^{-\mathrm{i}r\sigma^\pm}\,, & & \\
\text{closed:} & \psi^\mu_- = \dfrac{1}{\sqrt{2}}\sum_r \psi^\mu_r \mathrm{e}^{-\mathrm{i}2r\sigma^-}\,, & & \psi^\mu_+ = \dfrac{1}{\sqrt{2}}\sum_r \tilde\psi^\mu_r \mathrm{e}^{-\mathrm{i}2r\sigma^+}\,, \\
\text{where} & r\in\mathbb{Z}+\nu \quad \text{with} & R\ :\ \nu=0\,, & NS\ :\ \nu=\tfrac12\,.
\end{array} \tag{4.9}$$

As mentioned, similarly as in the bosonic case, there are the constraints. They arise from vanishing of energy-momentum components T_{++} (the L_n) and T_{--} (the $\tilde L_n$) and there are now also similar supercurrents J_+ and J_- with components are denoted[3] as G_r (and $\tilde G_r$ for closed modes).

4.3 Quantization

Identification of momenta leads to quantization of the modes in operators with canonical commutation relations. For fermions, these are in fact anticommutation relations (with the notation $[a,b]_+ = ab+ba$)

$$[\psi^\mu_r,\psi^\nu_s]_+ = \eta^{\mu\nu}\delta_{r+s}\,. \tag{4.10}$$

This defines again Fock spaces with the negative modes as creation operators and the positive ones as annihilation operators. But there is one exception that one has to treat specially: the ψ_0 mode (for the R case). Their anticommutation relations are similar to the 'gamma matrices' of a Clifford algebra. Therefore the vacuum can not be a single state, but should be a spinor on which these gamma matrices can act. This will give rise to the fermions in the spectrum (in spacetime) as we will see below.

As mentioned, these modes contribute to the Virasoro generators, and to the analogous fermionic operators:

$$\begin{aligned}
L_m &= \tfrac12\sum_{n\in\mathbb{Z}} :\alpha^\mu_n\,\alpha^\mu_{m-n}: +\tfrac14\sum_{r\in\mathbb{Z}+\nu}(2r+m) :\psi^\mu_{-r}\psi^\mu_{m+r}: +a_\nu\delta_m\,,\\
G_r &= \sum_{n\in\mathbb{Z}}\alpha^\mu_{-n}\psi^\mu_{r+n} \qquad R\ :\ a_\nu=\tfrac1{16}D\,,\qquad NS\ :\ a_\nu=0\,.
\end{aligned} \tag{4.11}$$

There is again the problem of normal ordering. But you can see that this enters only for L_0. We could have defined here the generator L_0 without the constant a_ν. But this is chosen in order that the Virasoro generators satisfy the algebra as in (2.48) (see the footnote added to this equation).

Exercise 4.1: Check the consistency of these formulae: e.g. rename $r'=-(m+r)$ and see that this is consistent.

[3]Traditionally these were denoted G_r for NS and F_n for R, but this notation is more convenient.

Exercise 4.2: Calculate the Noether current of the supersymmetry (4.3) for the action (4.2) and show how this gives rise to the G_r and $\tilde{G}_r$ in (4.11). Some help may be needed: if the Lagrangian is invariant under a symmetry up to a total derivative, $\delta(\epsilon)\mathcal{L} = \partial_\alpha K^\alpha(\epsilon)$, then the Noether current is $J^\alpha(\epsilon) = \frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi}\delta(\epsilon)\phi - K^\alpha(\epsilon)$, where the first term involves a sum over all fields ϕ. Further it is useful to have the relations

$$\rho^+ = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad \rho^- = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}.$$

With similar methods as in the bosonic case, we can calculate the algebra of the currents. We obtain

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}\,, \\
[L_m, G_r] &= (\tfrac{1}{2}m - r)G_{m+r}\,, \\
[G_r, G_s]_+ &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s}\,,
\end{aligned} \tag{4.12}$$

Exercise 4.3: Check that the central terms of the fermionic generators are not independent of the bosonic terms, by considering a (super)Jacobi identity involving e.g. $G_{1/2}$, G_r and $L_{-r-1/2}$ and a similar one for the R case.

Exercise 4.4: Advanced: the generators $L_{\pm1}$, L_0 and $G_{\pm1/2}$ form a closed subalgebra. Which superalgebra is this?

The conclusion of the above exercise is that there are only two numbers that depend on the matter content: the central charge as in (2.48) and a constant for the normal ordering to determine the super Virasoro constraints on physical states:

$$L_n|\text{phys}\rangle = 0 \quad \text{for } n > 0\,, \qquad (L_0 - a)|\text{phys}\rangle = 0\,, \qquad G_r|\text{phys}\rangle = 0 \quad \text{for } r \geq 0\,. \tag{4.13}$$

These constraints are not independent. In the NS case, the odd L_n are squares of G_r and thus we have only to check the even ones when we have imposed the last constraint. In the R case, the even L_n are squares of fermionic ones. For L_0 we find $L_0 = G_0G_0 + c/24$. Therefore, $a = c/24$ for the R case. Note that there was no ordering ambiguity in G_0 so that we could not add a constant to the last constraint.

Explicit calculations give

$$c = \tfrac{3}{2}D\,. \tag{4.14}$$

In the bosonic case we found $c = D$, so we can say that the fermionic modes contributed each with a factor 1/2. We mentioned in the bosonic case that the ghosts contributed with $c = -26$, such that the anomaly-free condition lead to $D = 26$. The superghosts contribute with $c = 11$. One can then check that $D = 10$ is required to cancel the central term. This is the famous result that superstring theories live in 10 dimensions.

Exercise 4.5: Check that the fermionic additions to L_0 can still be interpreted as a number operator, such that

$$\begin{array}{lll} \text{open string} & : & L_0 = -\alpha' M^2 + N + a_\nu\,, \\ \text{closed string} & : & L_0 = -\tfrac{1}{4}\alpha' M^2 + N + a_\nu\,, \qquad \tilde{L}_0 = -\tfrac{1}{4}\alpha' M^2 + \tilde{N} + a_\nu\,. \end{array} \tag{4.15}$$

You should thus check that e.g. $[N, \psi_{-1/2}\psi_{-3/2}\alpha_{-1}] = 3\psi_{-1/2}\psi_{-3/2}\alpha_{-1}$.

4.4 Physical string modes

4.4.1 Open string modes and the GSO projection

Let us follow the method that we used for the bosonic modes: analyzing lowest modes. We start with the NS modes for open strings. The $N = 0$ mode is physical and has mass $\alpha' M^2 = -a$. Then at level $N = 1/2$ the modes are

$$|a, k, 1/2\rangle \equiv a_\mu \psi^\mu_{-1/2} |k\rangle\,. \tag{4.16}$$

the $G_{1/2}$ constraint is the only relevant one, and leads to an equation $k^\mu a_\mu = 0$. This looks thus similar to what we found for the $N = 1$ mode in the bosonic case. It looks like an equation for a massless vector. This gives already the indication that we should take $a = 1/2$, such that this mode is massless. That is confirmed by other more complete calculations.

For the $N = 1$ case we start in a way similar to the $N = 2$ of the bosonic case:

$$|a, b, k, 1\rangle \equiv a_\mu \alpha^\mu_{-1} + b_{\mu\nu} \psi^\mu_{-1/2} \psi^\nu_{-1/2} |k\rangle\,. \tag{4.17}$$

Observe, however, that $b_{\mu\nu}$ is now antisymmetric as the operators $\psi_{-1/2}$ are fermionic. The role of the constraint $L_1|\text{phys}\rangle = 0$ in that case is taken here by the constraint $G_{1/2}|\text{phys}\rangle = 0$. Nothing corresponds to $L_2|\text{phys}\rangle = 0$ in that case, as here the constraint $L_1|\text{phys}\rangle = 0$ is a consequence of $G_{1/2}|\text{phys}\rangle = 0$. Note, however, that in the bosonic case, this determined just the trace $b_\mu{}^\mu$, while this is absent here due to the antisymmetry. The norm gives here again that the states determined by a_{0i} are spurious if $a = 1/2$.

For the Ramond sector, as mentioned we should extend the vacuum such that it is a spinor representation of the Clifford algebra spanned by ψ^μ_0. In general a spinor representation in D dimensions has size $2^{\text{Int}[D/2]}$. So this would lead here to spinors with 32 components. Special properties of Clifford algebras occur with D mod 8. Thus here we have the same special case as in $D = 2$: we can split the spinor in chiral and antichiral parts, keeping both parts real. This will play an important role later. For now, let us indicate these Fock vacua states as $|k, \alpha\rangle = 0$, where the first entry determines the eigenvalue under p^μ and the index $\alpha = 1, \ldots, 32$ is the representation index of this spinor. The ψ^μ_0 operators are then the gamma matrices

$$\psi^\mu_0 \to \frac{1}{\sqrt{2}} (\Gamma^\mu)_\alpha{}^\beta\,. \tag{4.18}$$

The constraint for G_0 is in this case similar to a massless Dirac equation

$$(\not{k})_\alpha{}^\beta |k, \beta\rangle = 0\,. \tag{4.19}$$

Indeed, it turns out that this is the right interpretation. Hence we find for the Ramond case that with $N=0$ we should have $a=a_\nu=10/16=5/8$.

Let us now review what we have. We have a tachyon in the NS sector, and a massless vector in NS and a massless spinor in R. The spinor has 32 components before use of the field equations, but similarly as for the well-known analysis of the Dirac equation in 4 dimensions, only 1/2 of these are physical. This you can also infer from the remark in the first lecture that massless particles are representations of SO$(D-2)$. In this case the spinor representations of SO(8) are minimally 8-dimensional. That fits with the count: first $32/2=16$ by splitting in chiral versus antichiral parts, and then the field equations reducing it to $16/2=8$. However, so far we did not restrict to chiral spinors.

We did not find a supersymmetric spectrum in spacetime so far. We had supersymmetry on the worldsheet, but not between the physical states. The bosonic states are the ones from the NS sector. That has a tachyon but no supersymmetric partner, as there is no tachyon in the RS sector. In the massless sector there is a vector in the NS sector. This has 8 physical degrees of freedom. It could be in a supersymmetric multiplet with a spinor of the R sector if we project out one chiral fermion from the latter. In fact, supersymmetry knows a supersymmetric Maxwell (or even Yang–Mills) multiplet, consisting of a chiral spinor λ and a vector A_μ.

This leads to the GSO projection. We introduce a projection that acts on the NS and on the R sector. In the NS sector we write

$$NS\ :\ P=-(-1)^{\sum_{r=1/2}^{\infty}\psi^\mu_{-r}\psi^\mu_r}\,. \tag{4.20}$$

Clearly it squares to 1, so it is a projector. It says that any state with an even number of ψ_{-r} creations $P=-1$ while those with an odd number satisfy $P=1$. Thus imposing $P=1$ on physical states eliminates the tachyon. On the R side we define

$$R\ :\ P=\pm\Gamma_*(-1)^{\sum_{r=1}^{\infty}\psi^\mu_{-r}\psi^\mu_r}\,,\qquad \Gamma_*=\Gamma_0\Gamma_1\ldots\Gamma_9\,. \tag{4.21}$$

The overall sign is at random. At the massless mode the projector is thus $\pm\Gamma_*$ which selects only the chiral (antichiral) states. It is clear that at this point it is arbitrary whether the chiral or antichiral ones are taken.

So far, this projection seems something arbitrary. However, it turns out that such a projection is consistent with the interactions. That is something that we could not do for the bosonic string. At the quantum level it is even required for consistency. We will see more often that supersymmetry is a good guidance principle in string theory. Here, selecting the states that form supersymmetric representations eliminates the tachyon and leads to a consistent quantum theory.

4.4.2 Closed string modes

For the closed strings, the conditions for L_0 and $\tilde L_0$ lead to

$$\alpha' M^2=N-a_x=\tilde N-\tilde a_x\,,\qquad NS\ :\ a_x=\tfrac12\,,\qquad R\ :\ a_x=a-a_\nu=0\,. \tag{4.22}$$

We will at once also consider the GSO projection. This has to be applied in the left and the right sector separately. But now there is a subtlety in the R sector. The sign in (4.21) is arbitrary, but when there are two projections then the relative sign makes a difference. So we will have to distinguish between opposite sign for left and right projection ('*type A*') or same sign for left and right projection ('*type B*').

We have to discuss 4 sectors. Let us start by *the NS-NS sector*. We have at $N = \tilde{N} = 0$ a tachyon, but this is eliminated by the GSO projection. At the level $N = \tilde{N} = 1/2$ we find massless states, and it is very similar to the closed bosonic string. They are

$$\psi^{\mu}_{-1/2}\tilde{\psi}^{\nu}_{-1/2}|k\rangle \sim G^{(\mu\nu)} + B^{[\mu\nu]} + \eta^{\mu\nu}\Phi \,. \tag{4.23}$$

Hence we find again a graviton, an antisymmetric tensor and a dilaton. We will not consider the massive modes (which are very high mass due to the value of α'). In total these are $8 \times 8 = 64$ physical states

Then consider *the R-R sector*. Here the massless sector is at $N = \tilde{N} = 0$. So the states are bispinors $|k, \alpha, \beta\rangle$, where α denotes the degeneracy for the vacuum of the left sector, and β the degeneracy of the right sector. We need some knowledge about Clifford algebras or representations of orthogonal groups. The bispinors are bosons and decompose as representations under the Lorentz group with antisymmetric products of Gamma matrices (and a charge conjugation matrix $\mathcal{C}^{\alpha\beta}$). All states at this level can then be decomposed as

$$\sum_{a=0}^{10} G^{(a)}_{\mu_1\ldots\mu_a}\left(\mathcal{C}\Gamma^{\mu_1\ldots\mu_a}\right)^{\alpha\beta}|k, \alpha, \beta\rangle \,. \tag{4.24}$$

The remaining G_0 and $\tilde{G}_0$ constraints imply that

$$\not{k} G^{(a)}_{\mu_1\ldots\mu_a}\Gamma^{\mu_1\ldots\mu_a} = G^{(a)}_{\mu_1\ldots\mu_a}\Gamma^{\mu_1\ldots\mu_a}\not{k} = 0 \,. \tag{4.25}$$

This gives the constraints

$$k^{\mu_1} G^{(a)}_{\mu_1\ldots\mu_a} = k_{[\mu_0} G^{(a)}_{\mu_1\ldots\mu_a]} = 0 \,. \tag{4.26}$$

These are like the Bianchi and field equations of field strengths. The latter one implies that $G^{(a)}_{\mu_1\ldots\mu_a}$ are field strengths of potentials $C^{(a-1)}_{\mu_1\ldots\mu_{a-1}}$,

$$G^{(a)}_{\mu_1\ldots\mu_a} = a\partial_{\mu_1} C^{(a-1)}_{\mu_2\ldots\mu_a} \,. \tag{4.27}$$

When we impose the projection GSO projection, we have to project the $\mathcal{C}\Gamma$ between $\frac{1}{2}(1 \pm \Gamma_*)$ and its transpose. This leaves for the type A only

$$\left[G^{(0)}\mathcal{C}^{\alpha\beta} + G^{(2)}_{\mu\nu}\left(\mathcal{C}\Gamma^{\mu\nu}\right)^{\alpha\beta} + G^{(4)}_{\mu_1\ldots\mu_4}\left(\mathcal{C}\Gamma^{\mu_1\ldots\mu_4}\right)^{\alpha\beta}\right]|k, \alpha, \beta\rangle \,, \tag{4.28}$$

while for the type B we have

$$\left[G^{(1)}_{\mu}\left(\mathcal{C}\Gamma^{\mu}\right)^{\alpha\beta} + G^{(3)}_{\mu\nu\rho}\left(\mathcal{C}\Gamma^{\mu\nu\rho}\right)^{\alpha\beta} + G^{(5)}_{\mu_1\ldots\mu_5}\left(\mathcal{C}\Gamma^{\mu_1\ldots\mu_5}\right)^{\alpha\beta}\right]|k, \alpha, \beta\rangle \,, \tag{4.29}$$

where the $a = 5$ are further projected to self-dual tensors, i.e.

$$G^{(5)}_{\mu_1\ldots\mu_5} = \frac{1}{5!}\varepsilon_{\mu_1\ldots\mu_{10}} G^{(5)\mu_6\ldots\mu_{10}} \,, \tag{4.30}$$

Exercise 4.6: Prove these restrictions by using

$$(\Gamma_*)^T\mathcal{C} = -\mathcal{C}\Gamma_*\,, \qquad \Gamma^{\mu_1\cdots\mu_5} = \frac{1}{5!}\varepsilon_{\mu_1\ldots\mu_{10}}\Gamma_*\Gamma^{\mu_6\cdots\mu_{10}}\,. \tag{4.31}$$

This can be stated group theoretical as products of representations of SO(1, 9):

$$\text{type A} : 16\times\overline{16} = 1+45+210\,, \qquad \text{type B} : 16\times 16 = 10+120+126\,. \tag{4.32}$$

If we have to count the physical states, we have to consider the gauge potentials $C^{(a)}$ and write them as representations of SO(8). The $G^{(0)}$ does not contribute,[4] while the others give rise to

$$\text{type A} : 8\ (C^{(1)})\ +\ 56\ (C^{(3)})\,, \qquad \text{type B} : 1\ (C^{(0)})\ +\ 28\ (C^{(2)})\ +\ 35\ (C^{(4+)})\,, \tag{4.33}$$

where the + in $C^{(4+)}$ indicates that this field has a field strength restricted by self-duality according to (4.30). Thus in total we find in each case 64 physical states.

When we go to *the NS-R sector*, we will find fermions again, as the vacuum states are spinors for the right-hand side: $|k,\alpha\rangle$. We have massless states with $N=1/2$ and $\tilde N = 0$. They are of the form

$$b^\mu_{-1/2}|k,a\rangle \sim \psi_\mu + \Gamma_\mu\lambda\,. \tag{4.34}$$

In this equation we indicated that a vector-spinor, as a representation of the Lorentz group, splits in a Γ traceless part and the Γ-trace.

Exercise 4.7: Prove that if a vector-spinor ψ_μ is gamma-traceless, $\Gamma^\mu\psi_\mu = 0$ then its Lorentz-transformation is gamma-traceless too.

The GSO projection is $P = +1$ on the left-hand side and takes one chirality from the right-hand side. This means that we have $8\times 8 = 64 = 56+8$ physical states as representations of SO(8).

It is clear that *the R-NS sector* is the same, but with spinors of opposite chirality for type A and with the same chirality for type B. They give again 64 physical states.

We thus found in total 128 bosonic and 128 fermionic states. As for the bosonic string, there is an effective field theory that describes the interactions at lowest order in α'. Indeed, there are two supergravity theories in 10 dimensions with 32 supersymmetries, one with supersymmetries defined by a non-chiral spinor parameter (type A), and one with two chiral parameters of the same chirality (type B). The type A is the reduction from the 'ultimate supergravity theory' which lives in 11 dimensions. One can show that 11 dimensions is the maximum for a supergravity theory. That 11-dimensional theory is the most elegant one. For some 20 years it looked like a minor flaw in superstring theory that it is not related to this theory, but only to the 10-dimensional ones. We will see in section 6.4 that superstring *is* related to 11-dimensional supergravity, but that needs more steps.

For now it is already beautiful that we find exactly these two 10-dimensional supergravity theories. The ψ_μ in the NS-R and R-NS sectors are the gauge fields for the supersymmetries.

[4]It can be considered as a constant, related to a mass parameter in 'Romans theory'. We do not discuss such extensions here.

4.5 Five superstring theories

4.5.1 Open and unoriented strings

In the superstrings that we saw above, we had 2 gravitinos, each gauging a (chiral, 16-component) supersymmetry. Therefore, the above theories are called the type IIA or type IIB theories. There is one supergravity with *one* supersymmetry in 10 dimensions. Its content is a a graviton (35 physical fields), a dilaton (1 component), and a two-form gauge field $C^{(2)}$ (28 components) on the bosonic side. On the fermionic side there is one gravitino ψ_μ (56 components) and a 'dilatino' λ (8 components). The theory has thus 64+64 components. This theory can couple to the 10-dimensional matter multiplets that we mentioned in relation to the open superstring.

You may have guessed how we can get this theory from the above-mentioned theories: by projecting to unoriented strings. Remember that this eliminated also the antisymmetric tensor in the bosonic string. The same happens here. To have symmetry between left and right sectors according to (2.50) we better use the type B string to start with. One could also use type A, but it would amount then to something equivalent to a combined operation of going back to type B and the projection in type A theory.

Furthermore, the projection in the R-R sector takes from the bispinor only the graded symmetric, i.e. antisymmetric part. By counting (a symmetric 16×16 matrix should have 120 components) it is clear from (4.32) that we preserve just $G^{(3)}$, i.e. the gauge 2-form $C^{(2)}$. The NS-R and R-NS are related by the projection, and we thus remain also with 64 fermionic fields: one gravitino and one λ.

This theory then couples to open strings. The absence of anomalies leads to a similar constraint as the one that we mentioned at the end of section 3.9: the gauge group has to be SO(32) = SO($2^{D/2}$). There is thus a projection to unoriented strings as in (2.52) with $\epsilon = -1$. The fact that all anomalies then consistently cancel (gauge and gravitational anomalies lead to the same conditions), was a major breakthrough for string theory, see Video: part 5 of 2nd hour (anomaly cancellation) (7').

4.5.2 Heterotic string theories

The most remarkable construction is the one of the heterotic string theories. In this case, the right and the left sector of the string theory are different. The right-moving sector is taken from the type II superstring, whereas the left-moving sector is taken from the bosonic string. So the cancellation of the central charge for the left and right Virasoro algebra is obtained in a completely different way.

The theory is defined as a 10-dimensional theory. To do so, the left-handed sector is considered as 10 + 16 dimensions, where the 16 dimensions are compactified. They are compactified in a well-defined way, namely on a lattice that satisfies peculiar properties:

$$X^I \simeq X^I + w^I_{(i)}\,, \qquad I = 1, \ldots, 16\,. \tag{4.35}$$

The vectors $\vec{w}_{(i)} = (w^I_{(i)})$, $i = 1, \ldots, 16$ generate a sixteen dimensional lattice Γ_{16}. Modular invariance requires that Γ_{16} is an 'even self-dual lattice'. Without going in details, let me mention that these lattices are very exceptional. The lowest dimensional one is 8-dimensional and is the root lattice of E_8. Then there are two in 16 dimensions: these are the root lattices of $E_8 \times E_8$ and the weight lattice of SO(32) with one of its spinor representations, i.e. the weight lattice of $Spin(32)/\mathbb{Z}_2$. These are the ones that we need.

They lead to the two heterotic string theories. Note that both groups have 496 generators. It turned out that this was one of the requirements to have an anomaly-free theory.

The massless sector of the theories are the fields of $N = 1$, $D = 10$ supergravity coupled to super-Yang–Mills multiplets for $E_8 \times E_8$ or SO(32).

Exercise 4.8: (Advanced): Study chapter 3 of lecture notes of Jan Govaerts (1986) explaining first general issues on compactification of bosonic strings on lattices and then the spectrum of the heterotic string theories.

4.5.3 Summary of the 5 theories

	Type IIA	Type IIB	$E_8 \times E_8$ Heterotic	SO(32) Heterotic	Type I SO(32)
String Type	Closed	Closed	Closed	Closed	Open (+ closed)
Oriented?	Yes	Yes	Yes	Yes	No
$D = 10$ Supersymmetry	$N = 2$ (non-chiral)	$N = 2$ (chiral)	$N = 1$	$N = 1$	$N = 1$
$D = 10$ Gauge groups	none	none	$E_8 \times E_8$	SO(32)	SO(32)
R-R sector	$C^{(1)}C^{(3)}$	$C^{(0)}C^{(2)}C^{(4+)}$	none	none	$C^{(2)}$

The massless fields follow from the entries in this table. All theories have a graviton and a dilaton. Furthermore, there is the antisymmetric tensor $B_{\mu\nu}$ if the string is oriented. The supersymmetry indicates fermionic fields: gravitini ψ_μ and partner λ. The indication $N = 2$ implies that there are 2 copies of these. The gauge group indicates whether there is furthermore a set of gauge fields and partners 'gaugini' in the adjoint of the indicated group.[5] The R-R sector is indicated in the last row.

The indication about 'string type' is for open strings that we discussed up to now, i.e. with Neumann boundary conditions. We will see that open strings with Dirichlet boundary conditions are possible for all the string theories that have a R-R sector, as they are supported by these fields (except from a "Neveu-Schwarz fivebrane"). This will be clarified in the next lectures.

It is worthwhile to note that the $E_8 \times E_8$ heterotic string has historically been considered to be the most promising string theory for describing the physics beyond the Standard Model. It was discovered in 1985 by Gross, Harvey, Martinec, and Rohm and for a long

[5]One could write U(1) in the entry for type IIA as one of the R-R fields can be considered as a U(1) gauge field, but this has no corresponding gaugino, and this U(1) symmetry has the same status as p-form symmetries corresponding any other R-R field.

time it was thought to be the only string theory relevant for describing our universe. This is because the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group of the standard model can fit quite nicely within one of the E_8 gauge groups. The matter under the other E_8 would not interact except through gravity, and might provide a answer to the Dark Matter problem in astrophysics. Due to our lack of a full understanding of string theory, answers to questions such as how is supersymmetry broken and why are there only 3 generations of particles in the Standard Model have remained unanswered.

Video: part 8 of 2nd hour (5 superstring theories) (5') and part 2 of hour 3 (an embarrassing problem) (4').

5 Dualities

Are there so many different string theories?
How are different string theories related?
Why do branes appear in superstring theories?
How can we treat the strong coupling effects of string theory?

We have seen five consistent distinct superstring theories: type I, types IIA and IIB, and heterotic SO(32) and $E_8 \times E_8$ string theories. There are even some more constructions, but we will not mention them here. The thinking was that out of these five candidate theories, only one was the actual correct Theory of Everything, and that theory was the theory whose low energy limit, with ten dimensions spacetime compactified down to four, matched the physics observed in our world today. The other theories would be nothing more than rejected string theories, mathematical constructs not blessed by Nature with existence.

But now it is known that this naive picture was wrong, and that the the five superstring theories are connected to one another as if they are each a special case of some more fundamental theory, of which there is only one. In the mid-nineties it was learned that superstring theories are related by duality transformations known as T duality and S duality. These dualities link the quantities of large and small distance, and strong and weak coupling, limits that have always been identified as distinct limits of a physical system in both classical and quantum physics. These duality relationships between string theories have sparked a radical shift in our understanding of string theory, and have led to the reasonable expectation that all five superstring theories – type I, types IIA and IIB, and heterotic SO(32) and $E_8 \times E_8$ – are special limits of a more fundamental theory.

In this chapter, we discuss these dualities that relate different theories. This will also lead us to the concept of 'D-branes'.

5.1 Supergravities

But to start with, we give an overview of supergravity theories, as these are the effective field theories of the superstring theories that we discuss and some knowledge of the possibilities and essential features is very useful.

We will consider the supergravities with fields that occur in the action up to two derivatives. This corresponds only to the first order in an α' perturbation theory of superstrings.

5.1.1 The list of supergravity theories.

To understand the possibilities for supersymmetry and supergravity you need two essential facts

1. List for any dimension the irreducible spinor representation.

2. A supergravity field theory has maximal supersymmetry with 32 real components (otherwise the multiplet would be too big to allow all states to be represented by field

theory). On the other hand a (rigid) supersymmetry field theory without gravity has maximally 16 real components.

E.g. in *4 dimensions*, a minimal spinor has 4 components. We have theories with $N = 1$ to $N = 8$. That means that there can be up to 8 supersymmetries (with 4 components each).

Any dimension larger than 11 has in its spinor representation at least 64 components (for Minkowski signature). Therefore the *maximal dimension for supergravity is 11.* The physical degrees of freedom reside only in 3 field: the graviton $g_{\mu\nu}$, a three-index antisymmetric tensor $A_{\mu\nu\rho}$ and a gravitino ψ_μ.

Exercise 5.1: Count the number of physical degrees of freedom using that these are massless fields. For the gravitino, remember the result of exercise 4.7, and use that a spinor in 9 dimensions has 16 components.

In *10 dimensions* there are 2 spinor representations of dimension 16: a chiral and an antichiral. Hence we have the following possibilities:

1. Supergravity based on 2 supersymmetries of opposite chirality, denoted as IIA.
2. Supergravity based on 2 supersymmetries of the same chirality, denoted as IIB.
3. Supergravity based on 1 supersymmetry, denoted as I.
4. Rigid supersymmetry: a vector multiplet with a vector A_μ and a spinor λ, each having 8 physical components. This one can couple to the supergravity I.

In *9 dimensions* a minimal spinor has 16 components, but, as in any odd dimension, any such representation is equivalent. Hence ...

5.1.2 Dimensional reduction

We consider here what happens when a theory in D dimensions is reduced to $D-1$ dimensions trivially. That is, we assume that the fields do not depend on one specific dimension, which we will denote as '9' for convenience.

An antisymmetric tensor like $C^{(a)}_{\mu_1\ldots\mu_a}$ splits in the lower dimensional Lorentz group in two parts. First, there is the part in which all indices of the tensor correspond to the lower dimensions. Hence this part is an a-form. Second, if one of the indices corresponds to the direction '9', then it reduces to an $a-1$ form.

Exercise 5.2: (elementary). Check that the field equation of electromagnetism, i.e. $\partial^\mu F_{\mu\nu} = 0$, reduces in one dimension lower to a field equation of a Maxwell field and that of a scalar field.

Exercise 5.3: (medium advanced). Show that a self-dual field strength in $2n$ dimensions implies that the field equations for the $n-1$ and $n-2$ form in dimension $2n-1$ vanish under the same conditions, hence only one of them is sufficient to describe the same physics.

Exercise 5.4: (elementary and useful to get acquainted with the effective theories of string theories). Consider the bosonic fields in the R-R sector of type IIA supergravity, as they were obtained in the context of the type IIA superstring in (4.33). Reduce these fields to 9 dimensions. Then consider the R-R fields of type IIB, and make the same exercise. Check that they fit in agreement with the fact that there is only one 9-dimensional supergravity in $D = 9$ as follows from the explanation at the end of section 5.1.1.

More details about the reduction of the NS-NS sector will follow in section 5.2.2.

5.2 T-duality for closed strings

5.2.1 Exchanging winding and momenta modes

The duality symmetry that obscures our ability to distinguish between large and small distance scales is called T-duality, and comes about from the compactification of extra space dimensions in a ten dimensional superstring theory. Let's take the X^9 direction and compactify it into a circle of radius R, so that

$$X^9 \approx X^9 + 2\pi R\,. \tag{5.1}$$

As the wave functions have factors $\mathrm{e}^{\mathrm{i}X\cdot p}$, a particle traveling around this circle will have its momentum quantized as n/R, where n is an integer. A particle in the nth quantized momentum state will contribute to the total mass squared of the particle as

$$m_n^2 = \frac{n^2}{R^2}\,. \tag{5.2}$$

Exercise 5.5: Check that the expansion for a massless scalar field in 10 dimensions,

$$\phi(x^M) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)\mathrm{e}^{\mathrm{i}nx^9/R}\,, \tag{5.3}$$

denoting $M = 0,\ldots,9$ and $\mu = 0,\ldots,8$, leads from a field equation $\partial_M\partial^M\phi = 0$ to a field equation $\partial_\mu\partial^\mu\phi_n = m_n^2\phi_n$.

A string can travel around the circle, too, and the contribution to the string mass squared is the same as above.

But a *closed string can also wrap around the circle*, something a particle cannot do. The number of times the string winds around the circle is called the winding number, denoted as w below, and w is also quantized in integer units. Tension is energy per unit length, and the wrapped string has energy from being stretched around the circular dimension. The winding contribution E_w to the string energy is equal to the string tension T_{string} times the total length of the wrapped string, which is the circumference of the circle multiplied by the number of times w that the string is wrapped around the circle.

$$E_w = 2\pi w R T_{\rm string} = \frac{wR}{\alpha'}\,. \tag{5.4}$$

The total mass squared for each mode of the closed string is therefore

$$m^2 = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}\Big(N+\tilde N - 2\Big)\,, \qquad N - \tilde N = n\,w\,. \tag{5.5}$$

These equations violate the level-matching condition mentioned above. Let us see in detail how this comes about. The first terms of the string solution have now an extra term compared to (2.19)

$$\begin{aligned}
X^9(\sigma,\tau) &= x^9 + 2\alpha' p^9\tau + 2wR\sigma + \tfrac12 {\rm i}\sqrt{2\alpha'}\sum_{n\neq 0}^{\pm\infty}\frac1n \alpha_n^9 {\rm e}^{-2{\rm i}n\sigma^+} + \tfrac12 {\rm i}\sqrt{2\alpha'}\sum_{n\neq 0}^{\pm\infty}\frac1n \tilde\alpha_n^9 {\rm e}^{-2{\rm i}n\sigma^-}\\
&= X_L^9(\sigma^+) + X_R^9(\sigma^-)\,,\\
X_L^9(\sigma^+) &= x_L^9 + \sqrt{2\alpha'}\alpha_0^9\sigma^+ + \tfrac12 {\rm i}\sqrt{2\alpha'}\sum_{n\neq 0}^{\pm\infty}\frac1n \alpha_n^9 {\rm e}^{-2{\rm i}n\sigma^+}\,,\\
X_R^9(\sigma^-) &= x_R^9 + \sqrt{2\alpha'}\tilde\alpha_0^9\sigma^- + \tfrac12 {\rm i}\sqrt{2\alpha'}\sum_{n\neq 0}^{\pm\infty}\frac1n \tilde\alpha_n^9 {\rm e}^{-2{\rm i}n\sigma^-}\,,
\end{aligned} \tag{5.6}$$

where $x^9 = x_L^9 + x_R^9$ and

$$\alpha_0^9 = \sqrt{\frac{\alpha'}{2}}p_L = \sqrt{\frac{\alpha'}{2}}\frac{n}{R} + \frac{1}{\sqrt{2\alpha'}}wR\,, \qquad \tilde\alpha_0^9 = \sqrt{\frac{\alpha'}{2}}p_R = \sqrt{\frac{\alpha'}{2}}\frac{n}{R} - \frac{1}{\sqrt{2\alpha'}}wR\,. \tag{5.7}$$

The constraint (2.46) now leads to the last of (5.5), while the first one follows from the sum of the L_0 and $\tilde L_0$ constraint.

The formulae (5.5) are invariant under the exchange

$$R \;\leftrightarrow\; \frac{\alpha'}{R}\,, \qquad n \;\leftrightarrow\; w\,. \tag{5.8}$$

In other words, we can exchange compactification radius R with radius α'/R if we exchange the winding modes with the quantized momentum modes.

This mode exchange is the basis of the duality known as *T-duality.* Notice that if the compactification radius R is *much smaller than the string scale* $\sqrt{\alpha'}$, then the compactification radius after the winding and momentum modes are exchanged is *much larger than the string scale.* So T-duality obscures the difference between compactified dimensions that are much bigger than the string scale, and those that are much smaller than the string scale.

One could in principle consider the region $0 \leq R \leq \sqrt{\alpha'}$, but it is easier to think in terms of a continuum in momentum modes rather than in winding modes.

The exchange takes

$$\alpha_0^9 \leftrightarrow \alpha_0^9, \quad \tilde\alpha_0^9 \leftrightarrow -\tilde\alpha_0^9 . \tag{5.9}$$

The full interacting theory contains phase factors like $\exp(\mathrm{i}p_L^9 X_L^9)$ and $\exp(\mathrm{i}p_R^9 X_R^9)$. Therefore the exchange should be generalized to

$$X_L^9(\sigma^+) \leftrightarrow X_L^9(\sigma^+), \qquad X_R^9(\sigma^-) \leftrightarrow -X_R^9(\sigma^-). \tag{5.10}$$

It turns out that then also all scattering amplitudes are invariant.

T-duality thus relates a theory compactified over a circle of radius R with the compactification over a circle of radius α'/R. Let us consider this now from the spacetime point of view.

5.2.2 Dimensional reduction in spacetime of the NS-NS sector

We consider the theory in D dimensions with 1 dimension compactified. For definiteness I will consider $D = 10$, but this is not essential. The compactified direction is denoted as 9. Further, I denote $M = 0, \ldots, 9$ and $\mu = 0, \ldots, 8$. The metric is assumed to depend only on the non-compactified coordinates, and is parametrized as

$$\mathrm{d}s^2 = G^{(10)}_{MN}\mathrm{d}X^M\mathrm{d}X^N = G^{(9)}_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu + \mathrm{e}^{2\sigma}(\mathrm{d}X^9 + A_\mu\mathrm{d}X^\mu)^2 . \tag{5.11}$$

Exercise 5.6: Check that this parametrization (observe that $G^{(10)}_{\mu\nu} \neq G^{(9)}_{\mu\nu}$) is such that under the remaining coordinate transformations with parameters ξ^μ and ξ^9 that only depend on X^μ, the fields $G_{\mu\nu}$, A_μ and σ behave as standard tensors, vector and scalar under the 9-dimensional GCT and that A_μ behaves as a gauge vector for ξ^9. This is the essential content of the Kaluza-Klein idea: one has produced an extra scalar, and a gauge vector describing electromagnetism.

The determinant is $G^{(10)} = G^{(9)}\mathrm{e}^{2\sigma}$ and the scalar curvature is (also this is not dependent on the value $D = 10$)

$$R^{(10)}(G^{(10)}) = R^{(9)}(G^{(9)}) - 2\mathrm{e}^{-\sigma}D_\mu\partial^\mu\mathrm{e}^\sigma - \tfrac14\mathrm{e}^{2\sigma}F_{\mu\nu}F^{\mu\nu} , \qquad F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]} . \tag{5.12}$$

The graviton-dilaton action included in (3.46) then reduces as follows after integration over X^9:

$$\begin{aligned} S &= \frac{1}{2\kappa_0^2}\int \mathrm{d}^{10}X\sqrt{G^{(10)}}\mathrm{e}^{-2\Phi}\left[R^{(10)}(G^{(10)}) + 4\partial_\mu\Phi\partial^\mu\Phi\right] \\ &= \frac{\pi R}{\kappa_0^2}\int \mathrm{d}^9X\sqrt{G^{(9)}}\mathrm{e}^{-2\Phi_9}\left[R^{(9)}(G^{(9)}) - \partial_\mu\sigma\partial^\mu\sigma + 4\partial_\mu\Phi_9\partial^\mu\Phi_9 - \tfrac14\mathrm{e}^{2\sigma}F_{\mu\nu}F^{\mu\nu}\right] , \end{aligned} \tag{5.13}$$

where $\Phi_9 = \Phi - \sigma/2$.

With the metric (5.11), the invariant length of the circle is $\rho = \mathrm{e}^\sigma R$. The effective string coupling is now

$$\mathrm{e}^{\Phi_9}(2\pi R)^{-1/2} = \mathrm{e}^{\Phi}(2\pi\rho)^{-1/2} . \tag{5.14}$$

The duality transformation transforms the real invariant length also as $\rho \leftrightarrow \alpha'/\rho$. This, and the invariance of the effective string coupling imply that the fields σ and Φ transform as

$$\sigma \leftrightarrow -\sigma\,, \qquad \mathrm{e}^{\Phi} \leftrightarrow \frac{\sqrt{\alpha'}}{\rho}\mathrm{e}^{\Phi}\,. \tag{5.15}$$

These transformations are part of a larger statement about the action of the T-duality on the background fields.

5.2.3 T-duality for the superstrings

The superconformal symmetry implies that the transformation (5.10) should be accompanied by a similar transformation on the $\tilde{\psi}^9_r$, i.e.

$$\psi^9_r \leftrightarrow \psi^9_r\,, \qquad \tilde{\psi}^9_r \leftrightarrow -\tilde{\psi}^9_r\,. \tag{5.16}$$

This in particular changes the sign of one of the Γ matrices [see (4.18)], or in the right sector, $\Gamma_* \leftrightarrow -\Gamma_*$. Therefore, type IIA and type IIB superstring theories are transformed in each other. This is in agreement with the fact that there is only one 9-dimensional supergravity with 32 supersymmetries (essentially because there is no chirality in odd dimensions). Both IIA and IIB supergravity reduce (over a circle) to the same 9-dimensional theory. That is the statement of their T-duality in supergravity.

Thus, T-duality relates type IIA superstring theory to type IIB superstring theory. Notice that a duality relationship between IIA and IIB theory is very unexpected, because type IIA theory has massless fermions of both chiralities, making it a non-chiral theory, whereas type IIB theory is a chiral theory and has massless fermions with only a single chirality.

T-duality is something unique to string physics. It's something point particles cannot do, because they don't have winding modes. If string theory is a correct theory of Nature, then this implies that on some deep level, the separation between large vs. small distance scales in physics is not a fixed separation but a fluid one, dependent upon the type of probe we use to measure distance, and how we count the states of the probe.

This sounds like it goes against all traditional physics, but this is indeed a reasonable outcome for a quantum theory of gravity, because gravity comes from the metric tensor field that tells us the distances between events in spacetime.

T-duality further relates heterotic SO(32) superstring theory to the heterotic $E_8 \times E_8$ superstring theory. We will skip the details on this duality here.

5.3 T-duality for open strings: D-branes

5.3.1 T-duality induces D8 branes

Let us now consider the $R \to 0$ limit of the open string spectrum. Open strings do not have a conserved winding around the periodic dimension and so they have no quantum number comparable to w, so something different must happen, as compared to the closed string case. In fact, it is more like field theory: when $R \to 0$ the states with nonzero momentum go to infinite mass, but there is no new continuum of states coming from winding. So we are left with a a theory in one dimension fewer. A puzzle arises when one remembers that theories with open strings have closed strings as well, so that in the $R \to 0$ limit the closed strings live in D spacetime dimensions but the open strings only in $D-1$.

This is perfectly fine, though, since the interior of the open string is indistinguishable from the closed string and so should still be vibrating in D dimensions. The distinguished part of the open string are the endpoints, and these are restricted to a $D-1$ dimensional hyperplane.

This is worth exploring in more detail. Write the open string mode expansion (2.5) as

$$\begin{aligned} X^\mu(\sigma,\tau) &= x^\mu + 2\alpha' p^\mu \tau + \mathrm{i}\sqrt{2\alpha'}\sum_{m\neq 0}^{\pm\infty}\frac{1}{m}\alpha^\mu_m \mathrm{e}^{-\mathrm{i}m\tau}\cos m\sigma = \\ X^\mu(\sigma^+,\sigma^-) &= X^\mu_+(\sigma^+) + X^\mu_-(\sigma^-)\ , \quad \text{where} \\ X^\mu_+(\sigma^+) &= \frac{x^\mu}{2} + \frac{x'^\mu}{2} + \alpha' p^\mu \sigma^+ + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_m \mathrm{e}^{-\mathrm{i}m\sigma^+}\ , \\ X^\mu_-(\sigma^-) &= \frac{x^\mu}{2} - \frac{x'^\mu}{2} + \alpha' p^\mu \sigma^- + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_m \mathrm{e}^{-\mathrm{i}m\sigma^-}\ , \end{aligned} \tag{5.17}$$

where x'^μ is an arbitrary number which cancels out when we make the usual open string coordinate. Imagine that we place X^9 on a circle of radius R. The T-dual coordinate is

$$\begin{aligned} X'^9(\sigma^+,\sigma^-) &= X^9_+(\sigma^+) - X^9_-(\sigma^-) \\ &= x'^9 + 2\alpha' p^9 \sigma - \sqrt{2\alpha'}\sum_{m\neq 0}\frac{1}{m}\alpha^9_m \mathrm{e}^{-\mathrm{i}m\tau}\sin m\sigma \\ &= x'^9 + 2\alpha'\frac{n}{R}\sigma - \sqrt{2\alpha'}\sum_{m\neq 0}\frac{1}{m}\alpha^9_m \mathrm{e}^{-\mathrm{i}m\tau}\sin m\sigma\ . \end{aligned} \tag{5.18}$$

Notice that there is no dependence on τ in the zero mode sector. This is where momentum usually comes from in the mode expansion, and so we have no momentum. In fact, since the oscillator terms vanish at the endpoints $\sigma = 0, \pi$, we see that *the endpoints do not move in the X'^9 direction!*. Instead of the usual Neumann boundary condition $\partial_\sigma X = 0$, we have $\partial_\tau X = 0$. More precisely, we have the Dirichlet condition that the ends are at a fixed place:

$$X'^9(\pi) - X'^9(0) = \frac{2\pi\alpha' n}{R} = 2\pi n R'. \tag{5.19}$$

In other words, the values of the coordinate X'^9 at the two ends are equal up to an integral multiple of the periodicity of the dual dimension, corresponding to a string that winds, see figure 4.

This picture is consistent with the fact that under T-duality, the definition of the normal and tangential derivatives get exchanged:

$$\begin{aligned} \partial_\sigma X^9(\sigma^+,\sigma^-) &= \partial_+ X^9_+(\sigma^+) - \partial_- X^9_-(\sigma^-) = \partial_\tau X'^9(\sigma^+,\sigma^-) \\ \partial_\tau X^9(\sigma^+,\sigma^-) &= \partial_+ X^9_+(\sigma^+) + \partial_- X^9_-(\sigma^-) = \partial_\sigma X'^9(\sigma^+,\sigma^-)\ . \end{aligned} \tag{5.20}$$

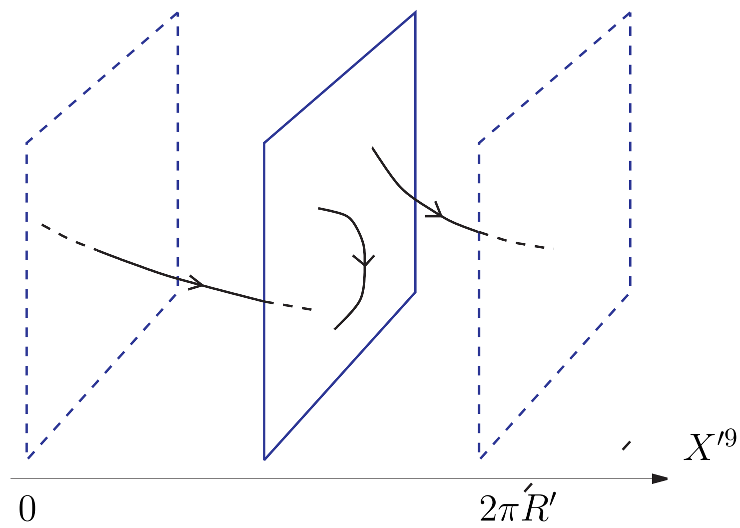

Figure 4: Open strings with endpoints attached to a hyperplane. The dashed planes are periodically identified. The strings shown have winding numbers zero and one.

Notice that this all pertains to just the direction which we T-dualized, X^9. So the ends are still free to move in the other 8 spatial dimensions, which constitutes a hyperplane called a "D-brane". Due to the 8 spatial directions, we shall denote it as a D8-brane.

We started here with showing that such D-brane solutions are induced by T-duality over a compact manifold. But we can also have these solutions in a large (non-compact) space. This is just the limit $R' \to \infty$. In that case, we should have $n = 0$ in (5.18).

We thus got in this way to the 'forgotten' possibility of boundary conditions for strings in (2.18): the Dirichlet boundary condition. We wrote there that the fluctuations of the brane at the endpoints vanish. Hence, there must be fixed endpoints, and we can write:

$$X^9(\tau, \sigma = 0) = x^9_{(1)}\,, \qquad X^9(\tau, \sigma = \pi) = x^9_{(2)}\,, \tag{5.21}$$

where $x^9_{(1)}$ and $x^9_{(2)}$ are fixed values. These we interpret as the positions of two D8 branes. The full solution with these Dirichlet boundary conditions is then

$$X^9(\sigma, \tau) = x^9_{(1)} + (x^9_{(2)} - x^9_{(1)})\frac{\sigma}{\pi} + \mathrm{i}\sqrt{2\alpha'}\sum_{m\neq 0}\frac{1}{m}\alpha^9_m \mathrm{e}^{-\mathrm{i}m\tau}\sin m\sigma\,. \tag{5.22}$$

5.3.2 Charged strings

Exercise 5.7: (important repetition of particle mechanics). Repeat first how charged objects are described in electromagnetism. A charged object is an addition to the action of the form

$$S_{\rm EM} = \int \mathrm{d}^4x\,(-\tfrac{1}{4})F_{\mu\nu}F^{\mu\nu} + \int_L \mathrm{d}t\,\left(\tfrac{m}{2}\dot{X}^\mu\dot{X}_\mu + q\dot{X}^\mu A_\mu\right)\,, \tag{5.23}$$

where the last integral is over the world-line L describing the history of the particle. Check that the Euler-Lagrange equations for A_μ give rise to Maxwell equations with a source. Consider e.g. that the particle is stationary at $\vec{X}=0$, and $X^0=t$. Check that the canonical momentum conjugate to $X^\mu(\tau)$ is

$$\Pi_\mu = \frac{\delta S}{\delta \dot{X}^\mu} = m\dot{X}_\mu + qA_\mu \,. \tag{5.24}$$

We have seen that open strings generate a background determined by a gauge vector A_μ. For open strings that have charges attached to their ends (Chan-Paton factors) it is a matrix, say $N\times N$. If this one is non-trivial, e.g.

$$A^9 = \frac{1}{2\pi R}\,\mathrm{Diag}(\theta_1,\ldots,\theta_N)\,, \tag{5.25}$$

it means that the background is not $\mathrm{U}(N)$ invariant, but broken to $\mathrm{U}(1)^N$ at worst (all values different).

The values (5.25) contribute to momenta similar to (5.24). For a string of unit charge with Chan-Paton factor i and j at both sides of the string, we should therefore replace n/R in the above T-duality transformation by

$$\frac{1}{2\alpha'}\frac{\partial X'^9}{\partial\sigma} = \frac{1}{2\alpha'}\frac{\partial X^9}{\partial\tau} = \frac{n}{R} + \frac{\theta_j-\theta_i}{2\pi R} + \ldots \,. \tag{5.26}$$

We obtain therefore equivalence with the general D-brane conditions if we put

$$x^9_{(2)} = \frac{\alpha'}{R}\theta_j\,, \qquad x^9_{(1)} = \frac{\alpha'}{R}\theta_i = x'^9\,. \tag{5.27}$$

Hence the distance between branes indicates the difference between several parts of the $\mathrm{U}(N)$ group that is described by the background. Thus this distance indicates the breakdown of the gauge group. When some eigenvalues are equal, it means that there is more than $U(1)^N$. If M branes coincide, this corresponds to a preservation of $\mathrm{U}(M)$ invariance. One can also notice that non-coinciding branes break the $\mathrm{U}(N)$ symmetry by writing down the mass formula where now the contribution of $(\dot{X}^9)^2$ is

$$M^2 = (p^9)^2 + \frac{1}{\alpha'}(N-1) = \left(n + \frac{\theta_j-\theta_i}{2\pi}\right)^2 \frac{R'^2}{\alpha'^2} + \frac{1}{\alpha'}(N-1)\,. \tag{5.28}$$

5.3.3 Dp-branes

We could repeat the procedure, which would bring us to a D7-brane. We state this as follows: with a Dp brane, the endpoints of the open string are fixed to a spacelike surface of dimension p. In terms of branes, what we usually call a membrane would be a two-brane, a string is called a one-brane and a point is called a zero-brane.

Thus we can write at the endpoints of the string (here written for $\sigma = 0$) the boundary conditions

$$\partial_\sigma X^\mu(0) = 0 \qquad \text{for } \mu = 0, 1, \ldots, p\,, \qquad X^\mu(0) = x'^\mu \qquad \text{for } \mu = p+1, \ldots, 9\,, \tag{5.29}$$

where the $9-p$ values x'^μ determine the position of the p brane in the 9-dimensional embedding space. If we apply this for $p = 9$, we see that the open string with Neumann boundary conditions can be interpreted as having its ends on a D9 brane. The 9-dimensional object fills the whole space.

In the example of a particle in exercise 5.7, we need a 1-form A to describe a charged world-line. This is like a D0-brane. In general for a p-brane we need therefore a $p+1$-form, i.e. $C^{(p+1)}$, that we can integrate over the $p+1$-dimensional world-volume of the brane:

$$S_p = \int \mathrm{d}^p X\, C^{(p+1)}\,, \qquad C^{(p+1)} \equiv C^{(p+1)}_{01\cdots p}\mathrm{d}X^0\mathrm{d}X^1 \ldots \mathrm{d}X^p\,. \tag{5.30}$$

In this way, we can understand the term with $B_{\mu\nu}$ in (3.43) as indicating that the string couples to this NS-NS 2-form.

A Dp brane is an object that carries one unit of charge under the R-R p-form field. Thus we see that D-branes are related to the presence of R-R sectors.

A magnetic charge would be an object where A in (5.23) would be replaced by a dual gauge field, i.e.

$$F_{\mu_1\mu_2} = 2\partial_{[\mu_1}A_{\mu_2]} \;\rightarrow\; \tilde{F}_{\mu_1\mu_2} = \tfrac{1}{2}\varepsilon_{\mu_1\mu_2\mu_3\mu_4}F^{\mu_3\mu_4} = 2\partial_{[\mu_1}\tilde{A}_{\mu_2]}\,. \tag{5.31}$$

The Bianchi identity and field equation for F translate to field equation and Bianchi identity for $\tilde{F}$. This interchanges electric and magnetic fields. The Maxwell equations are symmetric under this change apart from the fact that they normally do not include sources for the magnetic objects, as magnetic monopoles are not included in the theory.

In general dimensions, or with general forms, the magnetic dual is not a form of the same rank. E.g. when we would start the above procedure in 4 dimensions with a 2-form $A_{\mu\nu}$, with field strength $F_{\mu\nu\rho}$, the dual is a field strength of a scalar field $\tilde{F}_\mu = \partial_\mu\phi$. It is easy to convince you that in general dimension D, an n-form $A^{(n)}$ has as dual a $(D-n-2)$-form. Branes can be coupled 'electrically' or 'magnetically' to a form. As a p-brane couples with a $(p+1)$-form, it then follows that the dual of a p-brane is a $(D-p-4)$-brane.

One deduces from this that in type IIA and type IIB we have respectively the following branes

$$\text{IIA} \;:\; \text{D0, D2, D4, D6}\,, \qquad \text{IIB} \;:\; \text{D}-1\text{, D1, D3, D5, D7}\,. \tag{5.32}$$

The D-1 is in fact an instanton (no time direction either). The list could be completed by D8 (domain walls for type IIA) and D9 (space-filling brane in type IIB) that are not dynamical. This means that they are not associated to physical degrees of freedom, and that is why we did not find the R-R fields before.[6]

[6]In footnote 4 we mentioned $G^{(0)}$, which is dual to $G^{(10)}$ that supports the D8.

On the other hand, as a string couples to a 2-form B, the dual of the fundamental string, sometimes denoted as $F1$, is a 5-brane, denoted as $NS5$-brane.

The dynamics of the branes will be governed by the dynamics of the R-R fields. The configuration of a brane is a solution allowed by the supergravity theory, similar to the black hole as solution of GR. This will be discussed in section 6.2.

5.4 The full T-duality procedure

We now can take the limit to $R \to 0$, or $R' \to \infty$. This brings the dual picture back in flat space. In this case, only the mode $n = 0$ survives. These combined actions we will denote as a 'T-dualization'. Thus, a *T-dualization of one space direction* involves in principle the following steps:

1. Compactify that direction on a circle of radius R.
2. Rewrite in terms of R' and change the sign of the right-moving sector.
3. Take the limit $R' \to \infty$.

In the closed string part, we have seen that this transforms type IIA in type IIB theories.

A surprising revelation was that superstring theories are not just theories of 1-dimensional objects. There are higher dimensional objects in string theory with dimensions from 0 (points) to 9, called *Dp-branes*. By a T-duality in a direction perpendicular to a *Dp* brane, it is replaced by a D(p+1) brane. A T-duality in a direction tangent to the Dp brane changes it to a D($p-1$) brane. Considering the list in (5.32) we see that this is consistent with the fact that a T-duality relates type IIA to type IIB string theory.

Type I strings are the orientable strings. They were obtained before by performing a projections using the world-sheet parity operator Ω, that acts as

$$\Omega \;:\; X_+^\mu(\sigma^+) \leftrightarrow X_-^\mu(\sigma^-)\,. \tag{5.33}$$

When we combine this with a T-duality, then X_-^9 changes sign to determine the new coordinate X'. Therefore we get

$$X_+^{\prime 9}(\sigma^+) \leftrightarrow -X_-^{\prime 9}(\sigma^-)\,, \qquad \text{i.e.} \qquad X^{\prime 9}(\sigma^+,\sigma^-) \leftrightarrow -X^{\prime 9}(\sigma^-,\sigma^+)\,. \tag{5.34}$$

This is like a world-sheet parity combined with a $\mathbb{Z}_2$ symmetry around $X^{\prime 9} = 0$. This describes thus an 8-dimensional plane that we denote as an '*orientifold*' O8. Again this can be generalized to other dimensions, and the original construction to determine type I superstrings can be denoted as a construction using an orientifold O9.

5.5 S-duality

Some quantities in a theory cannot be calculated at all using perturbation theory, especially not for weak coupling. For example, the amplitude below cannot be expanded around the value $g = 0$:

$$A(g) = \exp(c/g^2)\,. \tag{5.35}$$

because the amplitude is singular there. This is typical of a tunneling transition, which is forbidden by energy conservation in classical physics and hence has no expansion around a classical limit.

String theories feature *two kinds* of perturbative expansions: an *expansion in powers of the string parameter* α' in the conformal field theory on the two-dimensional string world-sheet, and a *quantum loop expansion for string scattering amplitudes in D-dimensional space-time.* But unlike in particle theories, the string quantum loop expansion parameter is not just a number, but depends on one of the dynamic modes of the string, the vacuum expectation value of the dilaton field $\Phi(x)$, see (3.53). We will now look to transformations that change the value of this field, and hence of the coupling constant. This will *relate strong to weak coupling.* That is a very important possibility. It implies that the strong coupling behaviour of some theories can be obtained from a weak coupling expansion in another theory (sometimes the same one). We will especially study the self-S-duality of type IIB superstrings, and then comment on one example of S-duality between type I and heterotic SO(32) superstrings.

5.5.1 Self-duality in IIB

Let us start by type IIB string theory. Consider the massless bosonic fields. In the NS-NS sector we have $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. In the R-R sector there are the 0-form, 2-form and 4-form. A 0-form is in fact a scalar. We have already seen the effective action for the field Φ in (3.51) (in Einstein frame):

$$-\frac{1}{4\kappa^2}\int \mathrm{d}^{10}X\sqrt{g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\,. \tag{5.36}$$

The R-R scalar appears in string frame and Einstein frame as

$$-\frac{1}{4\kappa^2}\int \mathrm{d}^{10}X\sqrt{G}G^{\mu\nu}\partial_\mu C^{(0)}\partial_\nu C^{(0)} = -\frac{1}{4\kappa^2}\int \mathrm{d}^{10}X\sqrt{g}g^{\mu\nu}\mathrm{e}^{2\Phi}\partial_\mu C^{(0)}\partial_\nu C^{(0)}\,. \tag{5.37}$$

The definition (3.49) is here $g_{\mu\nu} = G_{\mu\nu}\mathrm{e}^{-\Phi/2}$. One can combine the two terms to

$$-\frac{1}{4\kappa^2}\int \mathrm{d}^{10}X\sqrt{g}g^{\mu\nu}\frac{\partial_\mu\lambda\partial_\nu\bar{\lambda}}{(\mathrm{Im}\,\lambda)^2}\,,\qquad \lambda\equiv C^{(0)}+\mathrm{i}\mathrm{e}^{-\Phi}\,. \tag{5.38}$$

This is similar to the integration measure that we encountered for the modular parameter τ in section 3.4. It shows an invariance under transformations

$$\lambda\to\frac{a\lambda+b}{c\lambda+d}\,,\qquad ad-bc=1\,. \tag{5.39}$$

This holds for all real $a,b,c,d\in\mathbb{R}$, forming the group $\mathrm{S}\,\ell(2,\mathbb{R})$. But in the full string theory, this group is broken to only $a,b,c,d\in\mathbb{Z}$, i.e. $\mathrm{S}\,\ell(2,\mathbb{Z})$, because there are objects with quantized charges that transform under this group.

Furthermore this symmetry rotates also $B_{\mu\nu}$ in $C^{(2)}_{\mu\nu}$. The action for these two fields is in Einstein frame

$$S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}X\,\sqrt{g}\left[-\tfrac{1}{12}\mathrm{e}^{-\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho}-\tfrac{1}{12}\mathrm{e}^{\Phi}\mathbf{G}_{\mu\nu\rho}\mathbf{G}^{\mu\nu\rho}\right], \tag{5.40}$$

with

$$H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}\,, \qquad \mathbf{G}_{\mu\nu\rho} = G_{\mu\nu\rho} - C^{(0)} H_{\mu\nu\rho}\,, \qquad G_{\mu\nu\rho} = 3\partial_{[\mu} C^{(2)}_{\nu\rho]}\,. \tag{5.41}$$

Exercise 5.8: First, prove that the power of Φ in the last term of (5.40) is again produced by going to Einstein frame. Then, prove that the action $S^{(2)}$ has this $\mathrm{S}\,\ell(2,\mathbb{R})$ invariance for

$$\begin{pmatrix} B_{\mu\nu} \\ C^{(2)}_{\mu\nu} \end{pmatrix} \rightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} B_{\mu\nu} \\ C^{(2)}_{\mu\nu} \end{pmatrix}, \tag{5.42}$$

using that the action can be written as proportional to (compare with (3.9))

$$\frac{1}{\operatorname{Im}\lambda} \begin{pmatrix} H_{\mu\nu\rho} & G_{\mu\nu\rho} \end{pmatrix} \begin{pmatrix} |\lambda|^2 & -\operatorname{Re}\lambda \\ -\operatorname{Re}\lambda & 1 \end{pmatrix} \begin{pmatrix} H^{\mu\nu\rho} \\ G^{\mu\nu\rho} \end{pmatrix}. \tag{5.43}$$

It is important to observe that the transformation $\lambda \rightarrow -\lambda^{-1}$ is e.g. for $C^{(0)} = 0$ the transformation $\Phi \rightarrow -\Phi$. According to (3.53) this means that strong coupling and weak coupling are interchanged.

Also this duality implies the D-branes. Indeed, a fundamental string carries charge under $B_{\mu\nu}$. As the S-duality relates this field to $C^{(2)}$, it implies that there must be other (solitonic) objects in type IIB that carry R-R charge. These are the D-branes.

5.5.2 type I - SO(32) heterotic

A second example is the duality between type I superstring and the heterotic SO(32) string. Their effective field theories have the same field content. In the heterotic string the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ appear nearly in the same way as in the NS-NS sector of the type II theories. On the other hand, in type I theory the $B_{\mu\nu}$ is eliminated by the Ω projection, but the 2-form $C^{(2)}$ is left from the R-R sector.

We may now look back to the end of the explanation of S-duality in type IIB. Then it is clear that these two effective actions are related by the transformation $\Phi \rightarrow -\Phi$, see (5.40) with $C^{(0)} = 0$ where the first term appears in the heterotic string, and the second one appears in IIA. Hence, the *weak coupling of the type I theory is related to the strong coupling of the heterotic theory* and vice versa.

Finally let us remark that there are many more dualities between compactified theories. E.g. consider the reductions of IIA theory on $K3 \times T^2$ ($K3$ is a 4-dimensional manifold with a complex structure and vanishing Ricci tensor) and heterotic theory on a 6-torus T^6. $K3$ breaks supersymmetry by a factor 1/2, i.e. we get from 32 to 16 supersymmetries. Hence both theories are 4-dimensional supergravities with $N = 4$ supersymmetry, which are dual. This duality has been studied a lot.

Video: part 3 of 3th hour (M theory) (6')

6 M-theory and Branes

What is M-theory?
Which branes exist in which theories?
What dimensions do they have?
Are branes charged and what is the tension of such branes?
Which are the symmetries of string theories?
Are branes consistent with a modified version of general relativity?
Are they related to other solutions of GR, as black holes?

In this lecture we are going to introduce the last links that relate string theory to an 11-dimensional theory, and view the whole picture of M-theory. An essential ingredient of all the dualities are the branes. Therefore, we will first study these branes from another perspective than in the previous lecture: from the spacetime point of view. Indeed, the branes are not just boundary conditions for strings. They are also dynamical objects with mass and charge. They occur as solutions in the effective supergravity theory.

The full theory is analogous to the action (5.23). The 3 terms get here the names

$$S = S_{\rm bulk} + S_{\rm DBI} + S_{\rm WZW}\,. \tag{6.1}$$

The first term, the 'bulk' action is the theory in spacetime. For the string theories discussed above, it is the supergravity theory in 10 dimensions. The other terms constitute the brane action. They are integrals over the $p+1$ dimensions of the p-brane worldvolume. The first one, 'Dirac-Born-Infeld (DBI) action' is like the kinetic term of the particle. The other one determines the interaction of the brane with the gauge fields giving charge to the brane. This is called the 'Wess-Zumino-Witten (WZW) action'.

An electromagnetic field configuration like the one of a point electric charge is a solution of the field equation for the bulk action, except in the point where the charge sits. There, a delta function source appears in the field equation, which can be understood as the contribution to the field equation from the last term. Indeed, then we write (e.g. taking the worldline as $\vec{X}=0$ and choosing the parametrization of the time on the worldline equal to the spacetime time coordinate)

$$q\int_L {\rm d}t\,\dot{X}^\mu A_\mu = q\int d^4x\,\delta^{(3)}(\vec{x})A_0\,. \tag{6.2}$$

This leads to the source term with $\delta^{(3)}(\vec{x})$ in the Maxwell equation that includes $\vec{\nabla}\cdot\vec{E}$.

Similarly, the branes will also be solutions of the bulk action apart from the singular planes where they are located. We will consider such solutions in supergravity, starting with the more familiar black hole solutions of GR. But we will stress immediately that the supersymmetry plays once more an important role. Therefore a short introduction to the supersymmetry algebra is useful.

6.1 Supersymmetry algebras

First, we clarify the relation between transformations and generators. We have already written supersymmetry transformations of fields, the first time in (1.10). These changes of fields are proportional to a parameter ϵ, and we can write, using spinor indices α,

$$\delta(\epsilon) = \epsilon^\alpha Q_\alpha \,, \tag{6.3}$$

i.e. the product of the parameter with an operation called the generator of the supersymmetry. This operation is for supersymmetry also a fermionic object, such that the elementary change of a field is of the same type as the field itself. When one calculates a commutator of two transformations, one obtains an anticommutator of the generators:

$$\begin{aligned} \delta(\epsilon_1)\,\delta(\epsilon_2) &= \epsilon_1^\alpha Q_\alpha \epsilon_2^\beta Q_\beta = \epsilon_2^\beta \epsilon_1^\alpha Q_\alpha Q_\beta \,, \\ [\delta(\epsilon_1),\,\delta(\epsilon_2)] &= \epsilon_2^\beta \epsilon_1^\alpha Q_\alpha Q_\beta - \epsilon_1^\alpha \epsilon_2^\beta Q_\beta Q_\alpha = \epsilon_2^\beta \epsilon_1^\alpha \left(Q_\alpha Q_\beta + Q_\beta Q_\alpha\right). \end{aligned} \tag{6.4}$$

The minimal supersymmetry algebra is realized e.g. on the scalar A in (1.10):

$$[Q_\alpha, Q_\beta]_+ = (\gamma^\mu \mathcal{C}^{-1})_{\alpha\beta} P_\mu \,. \tag{6.5}$$

Exercise 6.1: Check that with the normalizations given in (1.10) there is in fact a factor -2 missing in the right-hand side, where $P_\mu A = \partial_\mu A$. Some conventions that you may need are: a spinor is denoted as ϵ_α, and the components of the conjugate $\bar\epsilon = \epsilon^T \mathcal{C}$ are denoted as ϵ^α. For position of indices, write γ matrices as $\gamma^\mu{}_\alpha{}^\beta$, and $\mathcal{C}$ as $\mathcal{C}^{\alpha\beta}$, such that $\mathcal{C}^{-1}$ is written with indices as $\mathcal{C}^{-1}_{\alpha\beta}$. In 2,3,4 mod 8 dimensions (thus including also 10 and 11 dimensions), we can use charge conjugation and gamma matrices[7] such that $\mathcal{C}^T = -\mathcal{C}$ and $\Gamma_\mu^T = -\mathcal{C}\Gamma_\mu \mathcal{C}^{-1}$. This means that $\Gamma^{(n)}_{\mu_1\cdots\mu_n}\mathcal{C}^{-1}$ is symmetric if $n = 0,1$ mod 4, and antisymmetric for $n = 2,3$ mod 4. The matrix $\gamma_\mu\gamma_5$ that you will meet in the calculation can be written as the $n = 3$ case.

The supersymmetries commute with translations and are a spinor of Lorentz transformations:

$$[P_\mu, Q] = 0\,, \qquad [M_{\mu\nu}, Q] = -\tfrac{1}{4}\gamma_{\mu\nu} Q \,. \tag{6.6}$$

The extensions indicated by $N > 1$ mean that there are different supersymmetries Q^i with $i = 1, \ldots, N$. The minimal algebra in that case is

$$\left[Q^i_\alpha, Q^j_\beta\right]_+ = \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} \delta^{ij} P_\mu \,. \tag{6.7}$$

The generalizations that we want to discuss are called 'central charges'. The possibility of central charges is found in the classical work of Haag–Łopuszański–Sohnius in the context of $N = 2$ in 4 dimensions. In that case, (6.7) can be extended to

$$\left[Q^i_\alpha, Q^j_\beta\right]_+ = \left(\gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} \delta^i_j P_\mu + \varepsilon^{ij}\left[\mathcal{C}^{-1}_{\alpha\beta} Z_1 + (\gamma_5 \mathcal{C}^{-1})_{\alpha\beta} Z_2\right]. \tag{6.8}$$

[7]Gamma matrices in more than 4 dimensions are usually indicated as Γ_μ, rather than γ_μ, but one could use a uniform notation too.

The generators Z_1 and Z_2 commute with everything else and are thus really 'central'. They play an important role when looking for supersymmetric solutions of the theory. But the name 'central charges' has been generalized to include other generators that can appear in the anticommutator of supersymmetries. E.g. in $D = 11$ the properties of the spinors allow us to extend the anticommutator as

$$[Q_\alpha, Q_\beta]_+ = \left(\Gamma^\mu \mathcal{C}^{-1}\right)_{\alpha\beta} P_\mu + \left(\Gamma^{\mu\nu}\mathcal{C}^{-1}\right)_{\alpha\beta} Z^{(2)}_{\mu\nu} + \left(\Gamma^{\mu_1\cdots\mu_5}\mathcal{C}^{-1}\right)_{\alpha\beta} Z^{(5)}_{\mu_1\cdots\mu_5} \,. \tag{6.9}$$

The allowed structures on the right-hand side are determined by the symmetry of the matrices, see more in exercise 6.1. The 'central charges' Z are no longer Lorentz scalars, and thus do not commute with the Lorentz generators. They are therefore not 'central' in the group-theoretical meaning of the word, but play in the physical context the same role as the ones in (6.8), and therefore got the same name.

We will see below that the right-hand side of (6.8) implies that supersymmetric black holes (point particles) can have charges, and that (6.9) corresponds to the fact that the 11-dimensional theory has supersymmetric 2-branes and 5-branes.

6.2 BPS branes

In section 4, we had only closed strings in type II theories. However, we can postulate the existence of solitonic extended objects, such that there can be open D-strings with ends on these objects.

What makes a p-brane? A p-brane is a spacetime object that is a solution to the Einstein and other field equations in the low energy limit of superstring theory, with the energy density of the non-gravitational fields confined to some p-dimensional subspace of the 9 space dimensions in the theory. For example, in a solution with electric charge, if the energy density in the electromagnetic field was distributed along a line in spacetime, a moving point in space, this 1-dimensional line would be considered a p-brane with $p = 0$. We will first consider this $p = 0$ case in 4 dimensions: a solution of general relativity with mass and charge: a charged black hole.

6.2.1 BPS Black holes

The Schwarzschild black hole is a solution of GR, simply demanding the Ricci tensor to vanish: $R_{\mu\nu} = 0$. It is a non-perturbative effect in gravity. We do not get to black holes by perturbing around flat space.

Consider now an electrically charged black hole (Reissner-Nordstrøm (RN)) in 4 dimensions. If its mass is M and electric charge is q, the solution is

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1 - \frac{2M}{\rho} + \frac{q^2}{\rho^2}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{\rho} + \frac{q^2}{\rho^2}\right)^{-1} \mathrm{d}\rho^2 - \rho^2 \mathrm{d}\Omega^2 \,, \\ F &= \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = 2\frac{q}{\rho^2}\, \mathrm{d}t \wedge \mathrm{d}\rho \,, \qquad A = 2\frac{q}{\rho}\, \mathrm{d}t \,., \end{aligned} \tag{6.10}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ is the metric on a 2-sphere. It is a solution of the Einstein-Maxwell theory

$$S = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{g}\left[R(g) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] . \tag{6.11}$$

Two such black holes attract each other due to gravitational forces, and repel due to their electric charges. The total force between two such black holes i and j at a distance ρ_{ij} is

$$F_{ij} = -\frac{M_iM_j}{\rho_{ij}^2} + \frac{q_iq_j}{\rho_{ij}^2} . \tag{6.12}$$

If you consider black holes with $q_i = \pm M_i$ then the configuration is stable: the branes do not attract neither repel each other.

Another interesting feature is the following. There are 2 horizons at:

$$\rho_\pm = M \pm \sqrt{M^2 - q^2} \tag{6.13}$$

If $M < |q|$ the two horizons disappear and we have a naked singularity. For this reason in the context of classical general relativity the *cosmic censorship* conjecture was advanced that singularities should always be hidden inside horizon and this conjecture was formulated as the bound:

$$M \ \geq |q| . \tag{6.14}$$

Of particular interest are the black holes that saturate the bound. If $M = |q|$ the two horizons coincide and, setting:

$$M = |q| , \qquad \rho = r + M , \qquad r^2 = \vec{x}\cdot\vec{x} , \tag{6.15}$$

the solution can be rewritten as

$$\begin{aligned} ds^2 &= -dt^2\left(1+\frac{M}{r}\right)^{-2} + \left(1+\frac{M}{r}\right)^2\left(dr^2 + r^2\,d\Omega^2\right) \\ &= -\,H^{-2}(r)\,dt^2 + H^2(r)\,d\vec{x}\cdot d\vec{x} , \\ F &= 2dt\wedge dH^{-1} \qquad H \equiv 1 + \frac{M}{r} . \end{aligned} \tag{6.16}$$

$H(r)$ is a harmonic function in a three–dimensional space spanned by the three Cartesian coordinates $\vec{x}$ with the boundary condition that $H(r)$ goes to 1 at infinity.

These black holes can be considered as solutions of $N = 2$ supergravity. The latter has as physical fields a graviton, a gravitino, and a vector (called graviphoton). The configuration with (6.10) with zero gravitino solves all the field equations. It is interesting that the limit for cosmic censorship is a consequence of supersymmetry. In the supersymmetric theory appear only black hole solutions that verify this cosmic censorship bound.

The supersymmetry transformations of the bosons are proportional to fermions, hence in this configuration $\delta(\epsilon)$boson= fermion ϵ, is zero for these configurations for any ϵ. On the other hand, the transformation of the fermion is proportional to the boson configuration times ϵ. In fact it is

$$\delta(\epsilon)\psi_\mu^i = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{mn}\gamma_{mn}\right)\epsilon^i + \tfrac{1}{8}\varepsilon^{ij}\gamma^{\rho\sigma}F_{\rho\sigma}\gamma_\mu\epsilon^j\,. \tag{6.17}$$

This is non-zero for general configurations, and thus $\epsilon = 0$: there are no supersymmetries transformations left. However, when $M = q$, it turns out that the vanishing of the right-hand side restricts ϵ, but leaves 1/2 of the components free. The condition is a projection equation on the allowed supersymmetries.

Exercise 6.2: Prove the preserved supersymmetry. We use the conventions that a local frame is $e^m = e_\mu^m \mathrm{d}x^\mu$ and an index a indicates the local spacelike directions, i.e. $m = 0$ or a. Using the vierbeins $e^0 = H^{-1}\mathrm{d}t$ and $e^a = H\mathrm{d}x^a$, check that the spin connections, defined by $\mathrm{d}e^m + \omega^{mn}e^n = 0$ are

$$\omega^{0a} = -\omega^{a0} = -H^{-3}\partial_a H\mathrm{d}t\,, \qquad \omega^{ab} = 2H^{-1}\mathrm{d}x^{[a}\partial^{b]}H\,. \tag{6.18}$$

Due to the time-independence of the solution, we might expect solutions for the supersymmetries with a time-independent ϵ. Hence, it is easiest to check the $\mu = 0$ part of (6.17). One should obtain a projection operator that selects one half of the supersymmetries. (Hint: Do not forget that e.g. $\gamma^\mu = e_m^\mu \gamma^m$).

Those who want to check the full equation, should be aware that ϵ can depend (and does depend) on the spacelike directions. Hence, they should check the integrability condition $\partial_{[a}\delta\psi_{b]}$ to eliminate the derivative on ϵ, and obtain finally $\epsilon^i = H^{-1/2}\epsilon_0^i$, where ϵ_0^i is constant.

This is called a BPS solution. We saw 3 important properties:

1. saturated bound between mass and charge.
2. no forces between the configurations.
3. 1/2 supersymmetry is preserved.

One final warning: people use the terminology 1/2 supersymmetry preserved, by which they mean: from all the components of the local $\epsilon(x)$, there are 1/2 of the components zero, and the other half are constant.

6.2.2 Brane solutions

In section 4, we had only closed strings in type II theories. However, we can postulate the existence of solitonic extended objects, such that there can be open D-strings with ends on these objects.

In the brane theories, we have to consider also the dilaton. We have to consider a coupled system with a form field of the type

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{g}\left[R(g) - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2(n!)}\mathrm{e}^{a\Phi}F^{(n)}_{\mu_1\ldots\mu_n}F^{(n)\mu_1\ldots\mu_n}\right]. \tag{6.19}$$

The p-brane solutions are similar to what we saw for the RN black hole in 4 dimensions. They are of the form

$$\begin{aligned}
\mathrm{d}s^2 &= H^\alpha\left[-(\mathrm{d}x^0)^2+(\mathrm{d}x^1)^2+\ldots(\mathrm{d}x^p)^2\right]+H^\beta\left[(\mathrm{d}y^{p+1})^2+\ldots(\mathrm{d}y^9)^2\right]\,,\\
F &= \frac{2}{\Delta}\mathrm{d}H^{-1}\mathrm{d}x^0\mathrm{d}x^1\ldots\mathrm{d}x^p\quad(\zeta=1)\,,\\
&\qquad \text{or}\qquad F=\frac{2Q_{D-4-p}\tilde{d}}{\sqrt{\Delta}\,n!}\,\frac{y^{a_1}}{r^{D-d}}\mathrm{d}y^{a_2}\ldots\mathrm{d}y^{a_{D-d}}\varepsilon_{a_1\ldots a_{D-d}}\quad(\zeta=-1)\,,\\
\mathrm{e}^\Phi &= H^\gamma\,,\qquad r^2\equiv(y^{p+1})^2+\ldots+(y^9)^2\,.
\end{aligned}\tag{6.20}$$

We used general dimension D although we want to use it for $D=10$, but in this way you may even compare with the 4-dimensional example.

$$\begin{aligned}
&d=p+1\,,\qquad \tilde{d}=D-d-2\,,\qquad \Delta=a^2+\frac{2d\tilde{d}}{D-2}\,,\\
&\alpha=-\frac{4\tilde{d}}{\Delta(D-2)}\,,\qquad \beta=\frac{4d}{\Delta(D-2)}\,,\qquad \gamma=\frac{2\zeta a}{\Delta}.
\end{aligned}\tag{6.21}$$

Note that we denote the transverse directions with coordinates y^a where a runs over $D-d$ values. H is again a harmonic function in p directions, which is in general of the form

$$H=1+\frac{Q_p}{r^{\tilde{d}}}\,,\qquad \partial_a\partial^a H=0\,.\tag{6.22}$$

We gave 2 possibilities for F, corresponding to an electric ($\zeta=1$) or a magnetic ($\zeta=-1$) type of solution. This leads to 2 different relations between n and p

$$\zeta=1\ :\ n=p+2=d+1\,,\qquad \zeta=-1\ :\ n=D-2-p=\tilde{d}+1\,.\tag{6.23}$$

The value of Q_p will determine the charge and mass of the object.

Exercise 6.3: Check that the magnetic solution satisfies the Bianchi identity.

For the RR fields in 10 dimensions, we have $a=(5-n)/2$, such that $\Delta=4$. For the self-dual case in IIB, we have to add the electric and its dual magnetic value to determine $G^{(5)}$.

These solutions turn out to be invariant under half of the spacetime supersymmetry transformations (for even p in type IIA and odd p in IIB). In fact, their presence clearly reduces the Lorentz invariance to $\mathrm{SO}(1,p)\times\mathrm{SO}(9-p)$. But also the supersymmetry is reduced by having such objects.

For the type II theories, which have 2 supersymmetries of different or equal chirality one finds that $\epsilon^i(x)=g_{00}^{1/4}\epsilon_0^i$, where ϵ_0^i is constant and

$$\epsilon_0^2=\Gamma^0\Gamma^1\ldots\Gamma^p\epsilon_0^1\,.\tag{6.24}$$

Exercise 6.4: (nearly immediate): See that this is a consistent 1/2 projection only for even p in type IIA and odd p in IIB.

These solutions are thus 'BPS solutions'. One can see that for the consistency of such supersymmetric solutions, the algebra should have corresponding central charges. Indeed, as they are eliminated by some supersymmetry generator, we see that the contribution of the mass (the P_0 in a rest frame) in (6.5) should be cancelled by some other term. Therefore we need central charges in the algebra, and these are related to charges of the gauge field that support the solution. Thus: analysis of the algebra can already teach us the possibilities of BPS solutions.

6.2.3 Tensions and charges

To determine charges and masses of such a solution, one has to use Gauss' law flux integral at transverse infinity. The electric currents j_e are defined by the equation in form language

$$\mathrm{d}\,{}^*F = {}^*j_e\,. \tag{6.25}$$

Thus j_e is a d-form. Electric charges are integrals over the transverse space of the dual of this form. If $\mathcal{M}$ denotes the transverse space, with boundary $\partial\mathcal{M}$, Stoke's theorem determines the electric charge as

$$e_p = \int_{\mathcal{M}} {}^*j_e = \int_{\partial\mathcal{M}} {}^*F\,. \tag{6.26}$$

One can check that the field strengths at large distances fall off like $r^{-(\tilde{d}+1)}$. The transverse space has dimension $D-d=\tilde{d}+2$. Thus its boundary is a sphere of dimension $\tilde{d}+1$. We can thus integrate over this sphere. The volume of an n-dimensional sphere is

$$V_n = \omega_n r^n\,,\qquad \omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}\,. \tag{6.27}$$

We write the constant Q_p in the harmonic function in the form

$$Q_p = N_p c_p\,,\qquad c_p = \frac{2\sqrt{\pi}\kappa}{\tilde{d}_p\,\omega_{\tilde{d}_p+1}}(4\pi^2\alpha')^{(3-p)/2}\,, \tag{6.28}$$

where N_p will turn out to be an integer.

We define a charge that has the dimension of a tension as

$$\hat{Q}_p = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} {}^*F^{(p+2)}\,, \tag{6.29}$$

and find

$$\hat{Q}_p = N_p\mu_p g_S^{-1}\,,\qquad \mu_p = \frac{\pi^{1/2}}{\kappa_0}(4\pi^2\alpha')^{(3-p)/2}\,. \tag{6.30}$$

As we have infinite branes, the solution has in fact infinite mass. However, the tension τ_p (energy per world volume) is finite. There is an 'ADM' construction in general relativity for the tension. It can be extracted from the metric (in Einstein frame) in the g_{00} component.

$$g_{00} = -1 + \frac{2\kappa^2 \tau_p}{(D-2)\omega_{\tilde{d}+1} r^{\tilde{d}}} \,. \tag{6.31}$$

This determines τ_p in terms of N_p. We get

$$\tau_p = N_p \frac{\pi^{1/2}}{\kappa} (4\pi^2\alpha')^{(3-p)/2} \,. \tag{6.32}$$

We find the equality between mass and charge as $\tau_p = \hat{Q}_p$. More general black p-branes that do not necessarily satisfy the Bogomol'nyi bound have

$$\tau_p \geq \hat{Q}_p \,. \tag{6.33}$$

We will further consider elementary branes with $N_p = 1$.

The magnitude of the charges is of course determined by the brane terms in (6.1). We do not have the time to go in detail through the brane actions. The form is

$$\begin{aligned} S_{\mathrm{DBI}} &= -T_p \int \mathrm{d}^{p+1}\xi \, \mathrm{e}^{-\Phi} \sqrt{\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} \,, \\ S_{\mathrm{WZW}} &= \frac{\mu_p}{g_S} \int \mathrm{d}^{p+1}\xi \sum_n C^{(n)} \mathrm{e}^{B+2\pi\alpha' F} = \frac{\mu_p}{g_S} \int \mathrm{d}^{p+1}\xi \, C^{(p+1)} + \ldots \,. \end{aligned} \tag{6.34}$$

Here $T_p = \tau_p g_S$ and G_{ab} is the pull-back of $G_{\mu\nu}$:

$$G_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi)) \,. \tag{6.35}$$

F_{ab} is the (Yang-Mills) field strength on the brane, produced by the charges of the strings that are attached. The exponential in the WZW action should be understood as leaving only the terms that are $(p+1)$-forms. The relevant term for the future is the one mentioned in the last expression. These actions have a lot of symmetries.

In order that they are BPS, one should have $\mu_p = T_p$.

6.2.4 Dirac charge quantization

We are of course studying a quantum theory, and so the presence of both magnetic and electric sources of various potentials in the theory should give some cause for concern. We should check that the values of the charges are consistent with the appropriate generalization of the Dirac quantization condition.

Let us remember first the case in electromagnetism. There, a magnetic charge means that the integral of F over a surface surrounding it is non-zero. We can write

$$\mu_{\mathrm{m}} = \int_{S^2} F^{(2)} \,. \tag{6.36}$$

This shows that we cannot have everywhere $F^{(2)} = \mathrm{d}A^{(1)}$. If there is a particle of electric charge μ_e, then its path integral uses the action and when we make a closed loop P, it contributes

$$\exp\left(\frac{\mathrm{i}}{\hbar}\mu_\mathrm{e}\oint_P A^{(1)}\right) = \exp\left(\frac{\mathrm{i}}{\hbar}\mu_\mathrm{e}\int_D F^{(2)}\right), \tag{6.37}$$

where we have used again Stokes' law with D a two-dimensional surface with the path $P = \partial D$ as boundary. We can, however, take the boundary from both sides around the magnetic charge, and this should give the same result. Hence the integral over a full sphere should give a unit factor in the path integral. This leads to

$$\mu_\mathrm{e}\mu_\mathrm{m} = 2\pi n\hbar\,, \tag{6.38}$$

for some integer n.

A p-brane and a $(6-p)$-brane are sources for the dual field strengths. Therefore the same applies. A $(6-p)$ source is surrounded by a $(p+2)$-sphere. We should now have in the exponent of the path integral (with the conventions in (6.34))

$$\mu_p g_S^{-1}\oint A^{(p+1)} = \mu_p g_S^{-1}\int F^{(p+2)} = \mu_p g_S^{-1}\int {}^*F^{(8-p)} \tag{6.39}$$

Using the convention (6.29) we should have for elementary branes

$$\mu_p g_S^{-1} 2\kappa^2 \hat{Q}_{6-p} = 2\kappa_0^2\mu_p\mu_{6-p} = 2\pi\,. \tag{6.40}$$

This is verified with (6.30), proving that indeed the numbers N_p have to be integers.

While this argument does not apply directly to the case $p = 3$, as the self–dual 5–form field strength has no covariant action, the result follows by T–duality.

We summarize here the normalizations that are commonly used:

$$\begin{aligned}
&\tau_p = \frac{\pi^{1/2}}{\kappa}(4\pi^2\alpha')^{(3-p)/2}\,, \qquad T_p = \mu_p = \tau_p g_S = \frac{\pi^{1/2}}{\kappa_0}(4\pi^2\alpha')^{(3-p)/2}\\
&\kappa = \sqrt{8\pi G_N} = \kappa_0 g_S\,, \qquad g_S = \mathrm{e}^{\Phi_0}\,,\\
&T_\mathrm{string} = \frac{1}{2\pi\alpha'}\,,\\
&g_{\mu\nu} = \mathrm{e}^{(\Phi_0-\Phi)/2}G_{\mu\nu}\,,\\
&g^2_{\mathrm{YM},p} = \tau_p^{-1}(2\pi\alpha')^{-2}\,.
\end{aligned} \tag{6.41}$$

The latter is the coupling constant of the lowest order Dirac-Born-Infeld action, which is a Yang-Mills theory.

Exercise 6.5: Check the latter by identifying the constant in front of $F_{\mu\nu}F^{\mu\nu}$ in this lowest action from (6.34) as $-1/(4g^2_{\mathrm{YM},p})$.

Sometimes a further identification between κ_0 and α' is made, by choosing the normalization of the dilaton field, which determines the string coupling constant g_S. Choosing $g_S = T_{\mathrm{string}}/\tau_1$, we get

$$\begin{aligned}
&2\kappa_0^2 = \alpha'^4(2\pi)^7 \\
&\tau_p = (\alpha')^{-(p+1)/2}(2\pi)^{-p} g_S^{-1} \,, \\
&g^2_{\mathrm{YM},p} = (2\pi)^{p-2}\alpha'^{(p-3)/2} g_S \,.
\end{aligned} \tag{6.42}$$

6.3 The 11th dimension

Before string theory won the full attention of the theoretical physics community, the most popular unified theory was an 11-dimensional theory of supergravity. The 11-dimensional spacetime was to be compactified on a small 7-dimensional sphere, for example, leaving 4 spacetime dimensions visible to observers at large distances.

This theory didn't work as a unified theory of particle physics, because it doesn't have a sensible quantum limit as a point particle theory. But this 11-dimensional theory would not die. It eventually came back to life in the strong coupling limit of superstring theory in 10 dimensions.

How could a superstring theory with 10 spacetime dimensions turn into a supergravity theory with 11 spacetime dimensions? We've already learned that duality relations between superstring theories relate very different theories, equate large distance with small distance, and exchange strong coupling with weak coupling. So there must be some duality relation that can explain how a superstring theory that requires 10 spacetime dimensions for quantum consistency can really be a theory in 11 spacetime dimensions after all.

Since we know that all string theories are related, and we suspect that they are but different limits of some more fundamental theory, then perhaps that more fundamental theory exists in 11 spacetime dimensions? These question bring us to the topic of M theory.

6.3.1 Strong coupling of IIA superstrings

We have seen that strong-weak dualities relate IIB theory to itself. But what happens with the IIA theory when the coupling is large? We know that non-perturbatively the theory contains also the D0, D2, D4, . . . branes. The D-brane tensions τ_p, see (6.42), have dimension $[\mathrm{mass}]^{(p+1)}$. Turning it to an effective mass, we thus consider

$$(\tau_p)^{1/(p+1)} \approx \alpha'^{-1/2} g_S^{-1/(p+1)} \,. \tag{6.43}$$

Thus, at strong coupling, the $p = 0$, i.e. the D0 particles have the lowest mass:

$$\tau_0 = \frac{1}{g_S\sqrt{\alpha'}} \,. \tag{6.44}$$

States with n D0 particles have a mass

$$n\tau_0 = \frac{n}{g_S\sqrt{\alpha'}} \,. \tag{6.45}$$

As the coupling becomes large, these become all massless. It looks rather as particles from a Kaluza-Klein tower (5.2), where one has made a compactification on a circle with radius

$$R_{10} = g_S\sqrt{\alpha'}\,. \tag{6.46}$$

Therefore the strong coupling of type IIA looks like the $R_{10} \to \infty$ limit of this, which is an 11-dimensional theory!!

Therefore, let us give it a try. We start from the 11-dimensional supergravity theory, and compactify on a circle of radius R_{10}. Now we can follow the procedure as in section 5.2.2. E.g. the metric of 11 dimensions should reduce to the metric in 10 dimensions, the dilaton and the $C^{(1)}$ field in the R-R sector. On the other hand, the 3-form $A_{\mu\nu\rho}$ can reduce to $C^{(3)}_{\mu\nu\rho}$ and $B_{\mu\nu}$.

Exercise 6.6: Check that this can work. From (5.12) we see that the curvature appears in the action (with the $\sqrt{G}$) proportional to e^{σ}, and the curvature of the 1-form with $\mathrm{e}^{3\sigma}$. These factors should be adapted to $\mathrm{e}^{-2\Phi}$ for the metric, see (3.46), and without such a factor for the R-R field strength (in string frame). Rescaling the metric with a factor $G_{\mu\nu} = \gamma G^{(10)}_{\mu\nu}$, the latter being the analogue of $G^{(9)}_{\mu\nu}$ in (5.12), prove that $\gamma = \mathrm{e}^{\sigma} = \mathrm{e}^{2\Phi/3}$. Now use these factors to reduce the field strength of the 3-index tensor in 11 dimensions. Just check the overall factors. All dilaton factors should cancel in the field strength kinetic term for $C^{(3)}_{\mu\nu\rho}$, while they should lead to the factor $\mathrm{e}^{-2\Phi}$ for the field strength of $B_{\mu\nu}$.

The constant in front of the action is easiest to compare in Einstein frame. We get

$$\kappa_{11}^2 = \kappa^2 2\pi R_{10} = \tfrac{1}{2}(2\pi)^8 g_S^3 \alpha'^{9/2}\,, \tag{6.47}$$

where (6.41) and (6.42) have been used. Defining for convenience an 11-dimensional mass parameter from the gravitational coupling constant

$$M_{11}^{-9} = \frac{2\kappa_{11}^2}{(2\pi)^8} = g_S^3 \alpha'^{9/2}\,, \tag{6.48}$$

we can express the string parameters in terms of the 11-dimensional mass parameter and compactification radius as

$$g_S = (M_{11}R_{10})^{3/2}\,, \qquad \alpha' = M_{11}^{-3} R_{10}^{-1}\,. \tag{6.49}$$

6.3.2 M-branes

In M theory, there are also extended objects, but they are called M-branes rather than D-branes. As we have a 3-form in 11 dimensions, we expect an M-brane which has 2 space dimensions. Indeed, there is such a solution of the 11-dimensional supergravity, and this is called an M2 brane. The dual of this is according to the general rules a brane with 11-2-4=5 space dimensions, i.e. the M5 brane. M-theory thus has a membrane M2 and M5-branes as solitons, but no strings.

How can we get the strings that we've come to know and love from this theory? We can compactify the 11-dimensional M-theory on a small circle to get a 10-dimensional theory. If we take a membrane with the topology of a torus and wrap one of its dimensions on this compact circle, the membrane will become a closed string! In the limit where the circle becomes very small we recover the fundamental string (1-brane) of type IIA superstring theory.

An easy check of the duality is that we can find an M-theory origin for the D-brane states unique to the IIA theory. Recall that the IIA theory contains D0, D2, D4, D6, D8-branes as well as the NS fivebrane. The following table gives some examples:

M-theory on circle	**IIA in 10 dimensions**
Wrap membrane on circle	IIA superstring
Shrink membrane to zero size	D0-brane
Unwrapped membrane	D2-brane
Wrap fivebrane on circle	D4-brane
Unwrapped fivebrane	NS fivebrane

It turns out that we can relate all the solutions. I give one further test. If these pictures are correct, then the D2 tension should be equal to the M2 tension, while the the string is a membrane wrapped around a compact dimension, and should feel the size of that dimension.

Indeed, when we write the tension of the D2 brane in terms of 11-dimensional quantities, it is

$$\tau_2 = (\alpha')^{-3/2}(2\pi)^{-2}g_S^{-1} = (2\pi)^{-2}M_{11}^3\,. \tag{6.50}$$

Indeed, this tension is not dependent on the compactification radius. On the other hand

$$T_{\text{string}} = \frac{1}{2\pi\alpha'} = 2\pi R_{10}\tau_2\,, \tag{6.51}$$

exactly what is expected from wrapping the above brane over the circle!

We still don't know the fundamental M theory, but a lot has been learned about the 11-dimensional M theory and how it relates to superstrings in 10 spacetime dimensions.

6.4 The theory currently known as M

Technically speaking, M theory is is the unknown 11-dimensional theory whose low energy limit is the supergravity theory in 11 dimensions discussed above. However, many people have taken to also using M theory to label the unknown theory believed to be the fundamental theory from which the known superstring theories emerge as special limits.

When we collect the dualities that are known between string theories and the M-theory, we get figure 5:

The indication Ω refers to an orientifold addition, or projection to unoriented strings. There are two versions of the type I theory. The type I' is a theory containing open strings coupled to D-branes (in fact a solitonic sector of the type I string).

We can also get a consistent 10-dimensional theory if we compactify M-theory on a small line segment. That is, take one dimension (the 11-th dimension) to have a finite length.

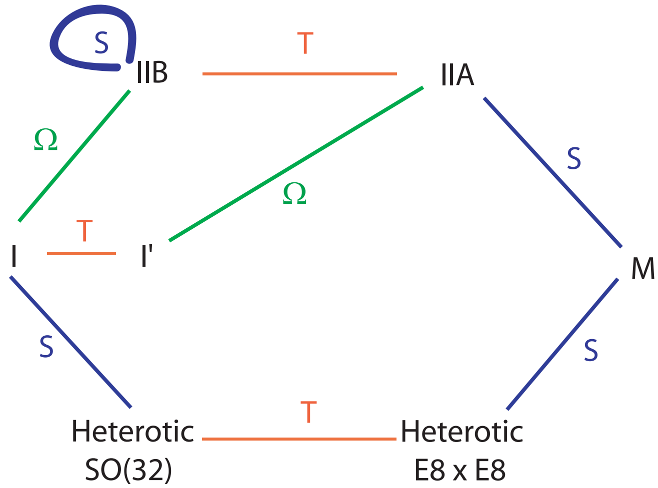

Figure 5: Dualities between string theories

The endpoints of the line segment define boundaries with 9 spatial dimensions. An open membrane can end on these boundaries. Since the intersection of the membrane and a boundary is a string, we see that the 9+1 dimensional worldvolume of the each boundary can contain strings which come from the ends of membranes. As it turns out, in order for anomalies to cancel in the supergravity theory, we also need each boundary to carry an E_8 gauge group. Therefore as we take the space between the boundaries to be very small we're left with a 10-dimensional theory with strings and an $E_8 \times E_8$ gauge group. This is the $E_8 \times E_8$ heterotic string! This is also known as the Horava-Witten theory.

It is important to realize that for this dualities to hold, the *solitonic solutions to the theories have to be included!*

With all these dualities, we suspect that there should be one single underlying theory. Some people denote this as M-theory, and would rather put '11-dimensional theory' on the right side in the figure. We still don't know what the fundamental theory behind string theory is, but judging from all of these relationships, it must be a very interesting and rich theory, one where distance scales, coupling strengths and even the number of dimensions in spacetime are not fixed concepts but fluid entities that shift with our point of view.

So given this new phase 11-dimensional phase of string theory, and the various dualities between string theories, we're led to the very exciting prospect that there is only a single fundamental underlying theory – M-theory. The five superstring theories and 11-D Supergravity can be thought of as classical limits. Previously, we've tried to deduce their quantum theories by expanding around these classical limits using perturbation theory. Perturbation has its limits, so by studying non-perturbative aspects of these theories using dualities, supersymmetry, etc. we've come to the conclusion that there only seems to be one unique quantum theory behind it all. This uniqueness is very appealing, and much of the work in this field will be directed toward formulating the full quantum M-theory.

7 Selected topics of string theory

What can be the form of the extra dimensions?
What are Calabi-Yau manifolds?
What is the AdS/CFT conjecture?
Has string theory something to say about black holes?
Has string theory something to say about the early universe?

7.1 Compactifications

The simplest way to make sense of a theory in higher dimensions is via compactifications. Let us start from a theory in 10 dimensions, and consider 6 dimensions to be compact. Hence we split the 10-dimensional coordinates X^M with $M = 0, \ldots, 9$ in x^μ, with $\mu = 0, 1, 2, 3$ and y^a, with $a = 1, \ldots, 6$.

We can then look (as simplest cases) whether the spacetime metric can be of the form

$$G_{MN} = \begin{pmatrix} \eta_{\mu\nu}(x) & 0 \\ 0 & G_{ab}(y) \end{pmatrix} . \tag{7.1}$$

This describes a flat 4-dimensional space, and $G_{mn}(y)$ is the metric of some compact space. Of course, a first requirement is that this metric solves the Einstein equation, which implies that it should have vanishing Ricci tensor.

The vacuum solution that we look for has zero spinors. To see whether such a solution preserves some supersymmetry we have then to check that the variations of the fermions are zero. Especially the variation of the ψ_a components of the gravitino ψ_M lead to interesting constraints on the geometry of the compact space. It is typically of the form

$$\delta\psi_a = D_a\epsilon = 0 . \tag{7.2}$$

This gives conditions on the y-dependence of the spinor. Depending on the geometry of the compact space, there may or may not be solutions of this equations. If there are solutions, then we can decompose the spinor in linear combinations of these solutions with x-dependent coefficients. The latter are the supersymmetries in 4 dimensions. E.g. a torus gives no restrictions on the spinors, and thus a compactification of a torus leaves as many supersymmetry components as there were in the large space. A Calabi-Yau manifold, which we will discuss below, breaks 3/4, or preserves 1/4 of the supersymmetries. If we thus start e.g. with a heterotic theory, that has $N = 1$, i.e. 16 real supersymmetry components in 10 dimensions, then it leaves 4 real components in 4 dimensions, i.e. $N = 1$ in $D = 4$. That is one of the reasons that these manifolds have been very popular for string compactifications (apart from preserving a E_6 factor of one of the E_8 groups in the heterotic theory, which is a suitable group for grand unification).

7.2 Calabi-Yau manifolds

Complex manifolds are manifolds with a complex structure. Without going into details, this means that the manifold is even-dimensional, and instead of $2n$ coordinates y^a, we can use z^α, and complex conjugates $\bar{z}^{\bar{\alpha}}$, where $\alpha = 1, \ldots, n$ and $\bar{\alpha} = 1, \ldots, n$. The affine connection on the manifold should preserve the complex structure J, i.e. the covariant derivative of J is zero. The complex structure is in these coordinates a matrix of the form

$$J_\alpha{}^\beta = \mathrm{i}\,, \qquad J_{\bar{\alpha}}{}^{\bar{\beta}} = -\mathrm{i}\,, \qquad J_\alpha{}^{\bar{\beta}} = J_{\bar{\alpha}}{}^\beta = 0\,. \tag{7.3}$$

Exercise 7.1: Prove that the vanishing of the covariant derivatives of J is equivalent to the requirement that the affine (and torsionless) connection has only non-zero components when its 3 indices are holomorphic, i.e. $\Gamma^\gamma_{\alpha\beta}$ or all antiholomorphic $\Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}$.

Calabi-Yau manifolds are complex manifolds with Ricci tensor which is exact, i.e. $R_{\alpha\bar{\beta}} = \partial_\alpha A_{\bar{\beta}} - \partial_{\bar{\beta}} A_\alpha$. Then there is a theorem conjectured by Calabi and proven by Yau that any such manifold has a unique Ricci-flat metric for a given complex structure and Kähler class (I will not further explain this here). For our purposes we need 3-folds (i.e. real dimensional 6, related to the reduction of 6 dimensions of the superstring).

To give you a flavour of what a Calabi-Yau manifold is, let me describe one.

A large number of CY can be described as hypersurfaces in a *weighted projective ambient space*. They correspond to loci of a complex *polynomial equation* $W(x) = 0$ where we identify $(x_1, \ldots x_5) \sim (\lambda^{w_1} x_1 \ldots \lambda^{w_5} x_{N+2})$ and $W(\lambda x) = \lambda^d W(x)$ with $\sum w_i = d$. There may be additional global identifications between points x.

Here is a concrete example. Take $w_1 = w_2 = 1$, and $w_3 = w_4 = w_5 = 2$. For these variables we make the following identifications

$$x_j \simeq \exp(n_j \frac{2\pi\mathrm{i}}{4})\, x_j \tag{7.4}$$

with

$$(n_1, n_2, n_3, n_4, n_5) \quad = \quad m_1(0,3,1,0,0) + m_2(0,3,0,1,0) + m_3(0,3,0,0,1)\,, \tag{7.5}$$

where $m_i \in \mathbb{Z}$. The polynomial is then of the form

$$W(x) = \frac{1}{8}(x_1^8 + x_2^8) + \frac{1}{4}(x_3^4 + x_4^4 + x_5^4) - \psi_0\, x_1 x_2 x_3 x_4 x_5 - \frac{1}{4}\psi_s\,(x_1 x_2)^4\,, \tag{7.6}$$

where ψ_0 and ψ_s can be chosen arbitrary (the other coefficients are normalizations).

The identifications remove one complex coordinate and the polynomial equation is also complex. Hence, we are left with a manifold with 3 complex coordinates.

Here, ψ_0 and ψ_s are 'moduli'. That is: they determine deformations of this surface. In the effective 4-dimensional theory they are scalar fields, and the vacuum is characterized by values of these fields.

This description is analogous to the description of a torus as follows. One starts from a 3-dimensional weighted projective complex space (Z, X, Y). The surface is defined as the points of this space where a weighted-homogeneous holomorphic polynomial vanishes. Due to the homogeneity we can view this in the weighted projective space, as a 1 complex dimensional surface, which is the Riemann surface. For genus 1, one can use the polynomial

$$0 \; = \; \mathcal{W}(X,Y,Z;u) \; = \; -Z^2 + \frac{1}{4}\left(X^4 + Y^4\right) + \frac{u}{2}X^2Y^2 \; , \tag{7.7}$$

in the projective space where Z has weight 2, and X and Y have weight 1. There is one complex parameter (modulus) u, which is related to the τ that we used before.

In fact, the T^2 torus is the CY 1-fold. The CY 2-fold is the K3 surface or the 4-torus T^4, and those that we use here are mostly 3-folds. (There are also reducible CY 3-folds: T^6 and K3$\times T^2$. Recently also CY 4-folds are used describing the reduction from 11 to 3 dimensions.

7.3 Black hole entropy

Black hole physics has many aspects of great interest to physicists with very different cultural backgrounds. These range from astrophysics to classical general relativity, to quantum field theory in curved space–times, particle physics and finally string theory and supergravity. This is not surprising since black holes are one of the basic consequences of a fundamental theory, namely Einstein general relativity. Furthermore black holes have fascinating thermodynamical properties that seem to encode the deepest properties of a fundamental theory of quantum gravity. Central in this context is the Bekenstein–Hawking entropy:

$$S_{\rm BH} = \frac{k_B}{G_N \hbar}\,\frac{1}{4}\,{\rm Area}_H \tag{7.8}$$

where k_B is the Boltzmann constant, G_N is Newton's constant, $\hbar$ is Planck's constant and ${\rm Area}_H$ denotes the area of the horizon surface.

This very precise relation between a thermodynamical quantity and a geometrical quantity such as the horizon area is a puzzle that stimulated the interest of theoretical physicists for more than twenty years. Indeed a microscopic statistical explanation of the area law for the black hole entropy has been correctly regarded as possible only within a solid formulation of quantum gravity. Superstring theory is the most serious candidate for a theory of quantum gravity and as such should eventually provide such a microscopic explanation of the area law. Although superstrings have been around for more than twenty years, a significant progress in this direction came only recently, after the so-called second string revolution (1995). Indeed black holes are a typical non–perturbative phenomenon and perturbative string theory could say very little about their entropy: only non perturbative string theory can have a handle on it. Progresses in this direction came after 1995 through the recognition of the role of string dualities.

Starting from p-branes in ten dimensions one can obtain four-dimensional black holes by dimensional reduction. This can be used to compute the entropy of the black hole: one counts the number N of microstates, i.e., excitations of the system, which belong to the same macrostate, i.e., the same total energy, charge and angular momentum:

$$S = \log N\,. \tag{7.9}$$

One finds that the two entropies agree, $S = S_{\mathrm{BH}}$, which confirms that the D-brane picture correctly captures the microscopic degrees of freedom of the black hole.

Also other aspects of black holes, as Hawking radiation, have been computed and found to be in agreement. These were considered as spectacular successes of the ideas of dualities.

7.4 AdS/CFT

The Anti-de Sitter to Conformal Field Theory (AdS-CFT) correspondence relates D-dimensional gravitational backgrounds to $(D-1)$–dimensional field theories. One of the roots of this idea is the so-called holographic principle, which claims that the physics beyond the horizon of a black hole can be described in terms of a field theory associated with its horizon. The D-brane picture of black holes can be viewed as a realization of this idea.

The AdS-CFT correspondence is a consequence of the relation between D-branes as described by a world-volume theory and as solutions of supergravity. One has to consider a specific limit of the theory on the world-volume which makes that theory to a conformal field theory. The corresponding limit in the p-brane regime is the near horizon limit. For $D3$ branes near the horizon, we neglect in the harmonic function $H = 1 + k/r^4$ the first term and keep only the r^{-4} term. The geometry then takes the form $\mathrm{AdS}^5 \times S^5$.

Exercise 7.2: Check that for the D3 brane solution in this limit, the metric splits completely in a part depending on the coordinates $x^0, \ldots, x^3, r$ and a part $\mathrm{d}\Omega^2$. That the latter is a 5-sphere S^5 is obvious. Write down the metric on the first 5-dimensional space for later use. (You may also check that the same split occurs for M2 and M5 branes).

The anti-de Sitter space is the coset

$$AdS_d = \frac{\mathrm{SO}(d-1,2)}{\mathrm{SO}(d-1,1)}\,. \tag{7.10}$$

This means that we need a space with isometries in $\mathrm{SO}(d-1,2)$ and where every point is invariant under a $\mathrm{SO}(d-1,1)$ subgroup.

To obtain a space with AdS metric, we start from defining it as a submanifold of a $(d+1)$-dimensional space with a flat metric of $(d-1,2)$ signature (for convenience we take here $\mu = 0, \ldots, d-2$)

$$\begin{array}{rcccl} \mathrm{d}s^2 = & \mathrm{d}X^\mu \eta_{\mu\nu} \mathrm{d}X^\nu & - & \mathrm{d}X^+ \mathrm{d}X^- & \\ & (d-2,1) & + & (1,1) & \Rightarrow (d-1,2) \end{array} \tag{7.11}$$

The AdS space is the submanifold determined by the $\mathrm{SO}(d-1,2)$-invariant equation

$$X^\mu \eta_{\mu\nu} X^\nu - X^+ X^- + R^2 = 0\,. \tag{7.12}$$

On the hypersurface one can take several sets of coordinates. E.g. the coordinates $\{x^\mu, r\}$ are defined by

$$X^- = r\,, \qquad X^\mu = r x^\mu\,, \qquad X^+ = r x_\mu x^\mu + \frac{R^2}{r}\,. \tag{7.13}$$

The latter being the solution of (7.12), given the first two as parametrization.

Exercise 7.3: Calculate the metric on the surface as the pullback of the metric in the embedding space. Check that this matches with the result of the first part of the D3 brane metric calculated in exercise 7.2.

Note: The (anti-)de Sitter algebra appears in a curved space of constant curvature where the translations do not commute, i.e.

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu[\rho} M_{\sigma]\nu} - \eta_{\nu[\rho} M_{\sigma]\mu}\,, \qquad [P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu} P_{\rho]} \\ [P_\mu, P_\nu] &= \frac{1}{2R^2} M_{\mu\nu}\,. \end{aligned} \tag{7.14}$$

With the opposite sign for the last commutator we would have the 'de Sitter algebra'. As written it is the 'anti-de Sitter algebra'. And for $R \to \infty$, we recover the Poincaré algebra. Defining $M_{d\mu} = -M_{\mu d} = R\,P_\mu$ we have generators $M_{\hat\mu\hat\nu} = -M_{\hat\nu\hat\mu}$ with $\hat\mu = 0, \ldots, d$, and defining the metric to be $\eta_{\hat\mu\hat\nu} = \mathrm{diag}(- + \ldots + -)$ the algebra can be concisely written as

$$[M_{\hat\mu\hat\nu}, M_{\hat\rho\hat\sigma}] = \eta_{\hat\mu[\hat\rho} M_{\hat\sigma]\hat\nu} - \eta_{\hat\nu[\hat\rho} M_{\hat\sigma]\hat\mu}\,, \tag{7.15}$$

i.e. it is the algebra $\mathrm{SO}(d-1, 2)$. Note that every point is invariant under the rotations around this point, while not invariant under the translations. In this sense we can write (7.10).

We have seen that in the general D3 brane configuration the supersymmetry is broken with a factor 1/2, i.e. to 16 supersymmetries. However, in this near-horizon limit the supersymmetry is doubled (there are extra solutions for the spinors). Thus we have 32 supersymmetries.

The boundary of the region where there is the AdS geometry is the brane. The Yang-Mills theory on this brane (4-dimensional) is also conformal and supersymmetric. In fact it has also 32 supersymmetries. 16 of these are as normal supersymmetries, and 16 others are 'special supersymmetries' that are necessary to have a superconformal group, i.e. the combination of supersymmetry with conformal symmetry. We have mentioned in section 3.1 that $\mathrm{SO}(D, 2)$ is the conformal group in D dimensions. Thus here this is $\mathrm{SO}(4, 2)$, i.e. the same group that is the AdS symmetry in the bulk. The remaining $\mathrm{SO}(6)$ in the bulk corresponds to the 'R'-symmetry, i.e. the symmetry $\mathrm{SU}(4) \approx \mathrm{SO}(6)$ that acts on the $N = 4$ theory in the brane.

One finds a correspondence between fields $\phi(x_{(5)})$ of the bulk supergravity theory and operators $\mathcal{O}(\xi^{(4)})$ of the Yang-Mills theory on the boundary (brane). (Here $x_{(5)}$ are coordinates on the five-dimensional bulk $(x^0, \ldots, x^4, r)$ and $\xi^{(4)}$ are coordinates on the four-dimensional brane.) A quantitative version of the conjecture, due to Gubser, Klebanov, Polyakov and Witten states that correlation functions between these operators in the Yang-Mills theory are equal to classical scattering of the string states. Thus a full quantum theory of Yang-Mills theory is equivalent to a classical theory of superstrings (in practice: supergravity).

7.5 Cosmology

see powerpoint presentation.

7.6 Some final remarks

You got an introduction to some aspects of string theory (or should we call it M-theory?). There are so many beautiful aspects and structures that fit so nicely that there should be truth in it. However, we cannot be sure that experiments will still in our lifetime be able to show convincingly that this is the theory that unites all interactions. We hope that high-energy experiments in LHC in CERN (operational in 2007) will show us some signs, especially supersymmetry. Though this would not be a final proof of string theory, it would indicate that we are on the right track. But also on other fronts the theory may be successful. Cosmology is becoming a more viable field in which string theory may prove to be useful. Apart from these applications it leads also to the possibility to understand non-perturbative aspects in field theory.

The models that are studied for applications in phenomenology or cosmology make use of various brane configurations, with intersections of branes, fluxes on branes, ... It still needs a lot of investigations to see all possibilities, but we have learned a lot in the last few years. I hope that these lectures will be useful for you to be able to understand the enthusiasm that string theorists have.

Thanks to Joke, Bruno, Stijn, Cristian, Bert, Johan, Wojciech, Stefaan, David, Rob, Jan, Dirk, Dieter, Caroline, Jos and Lode for the many remarks, questions and helpful comments.

A Notations

$c = 1$, $\hbar = 1$, such that $[\ell] = [t] = 1/[m]$.

The metric $(- + + ... +)$ and more in general I use the $(+ + +)$ convention for curvatures in the Misner-Thorne-Wheeler classification of conventions (see red pages after cover).

I also use δ_n for the quantity that is 1 if $n = 0$ and 0 otherwise. This is traditionally indicated as $\delta_{n,0}$.

(Anti)symmetrization is done with weight one:

$$A_{[ab]} = \tfrac{1}{2}\left(A_{ab} - A_{ba}\right) \qquad \text{and} \qquad A_{(ab)} = \tfrac{1}{2}\left(A_{ab} + A_{ba}\right) . \tag{A.1}$$

B Resources for popular introductions, these lectures and further study

A useful text to find useful references is: Donald Marolf, *Resource Letter NSST-1: The Nature and Status of String Theory*, [arXiv:hep-th/0311044]. There you find references ordered by level, and for the more specialized ones, by subject.

See especially the Websites under part IIB and the books 33-38 (unfortunately, nr. 39 seems to be out of print).

But there are some other reviews that I used, and that are not mentioned there:

- T. Mohaupt, "Introduction to string theory," Lect. Notes Phys. **631** (2003) 173 (Bad Honnef 2002, Aspects of quantum gravity), [arXiv:hep-th/0207249].

- C. V. Johnson, "D-brane primer," proceedings of 'Strings, branes and gravity', TASI 99. Eds. Jeffrey Harvey, Shamit Kachru, Eva Silverstein, Singapore, World Scientific, 2001; [arXiv:hep-th/0007170]

- H. Ooguri and Z. Yin, "Lectures on perturbative string theories,", in *Fields, strings and duality*, Proceedings of TASI 96, Ed. Costas Efthimiou and Brian Greene, Singapore, World Scientific, 1997, [arXiv:hep-th/9612254].

- M. Haack, B. Körs and D. Lüst, "Recent developments in string theory: From perturbative dualities to M-theory," [arXiv:hep-th/9904033].

- R. Dijkgraaf, "Fields, strings and duality,", in 'Quantum symmetries', proceedings Ecole d'été de Physique Théorique, Les Houches, France ed. by A. Connes, K. Gawedzki, J. Zinn-Justin. Amsterdam, The Netherlands, North-Holland, 1998; [arXiv:hep-th/9703136]

- K. S. Stelle, "Domain walls and the universe," in 'ICTP conference on super five brane physics in 5+1 dimensions', Edited by M. Duff, et al., Singapore, World Scientific, 1999; [arXiv:hep-th/9812086]

Especially the first one has been used extensively to prepare these lectures.

As you have noticed some gamma-matrix knowledge is useful. I have written the following review in which section 3 treats this in detail: A. Van Proeyen, "Tools for supersymmetry," *Annals of the University of Craiova, Physics AUC* **9 (part I)** (1999) 1–48, [arXiv:hep-th/9910030]